Theoretical Microfluidics

MICRO-718

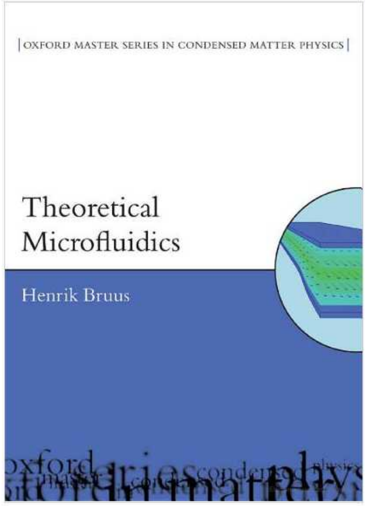
T. Lehnert and M.A.M. Gijs (EPFL-LMIS2)

EPFL - Lausanne
Doctoral Program in Microsystems and Microelectronics (EDMI)

The theoretical parts of this course are mainly based on:

Theoretical Microfluidics
Henrik Bruus

Oxford University Press, 2008 (Reprint 2010)
(ISBN 978-0-19-923509-4)



Equation/figure numbering in PART I of these lecture notes refers to the indicated edition of the book.

Numbering in PART II refers to an earlier edition !

All other references are indicated in the text.

Further reading:

J. Kirby, ***Micro- and nanoscale fluid mechanics: Transport in microfluidic devices***
Cambridge University Press 2010, ISBN: 978-0-521-11903-0

NT. Nguyen and S.T. Wereley, ***Fundamentals and applications of microfluidics***
Boston, Mass.: Artech House, 2006, ISBN: 1-58053-972-6

P. Tabeling, ***Introduction to microfluidics***
Oxford University Press 2005, ISBN: 0-19-856864-9

T.M. Squires and S.R. Quake, ***Microfluidics: Fluid physics at the nanoliter scale***
Reviews of Modern Physics, Vol. 77, pp. 977-1026 (2005)

“Theoretical” microfluidics and more…

EPFL

Course topics

Part I: *T. Lehnert* (EPFL-LMIS2)

1. Introduction
2. Governing equations and flow solutions
3. Microfluidic channels and circuits
4. Diffusion and mixing in microscale
5. Capillary effects and microdroplets

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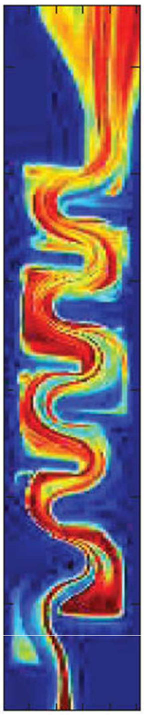
Course topics



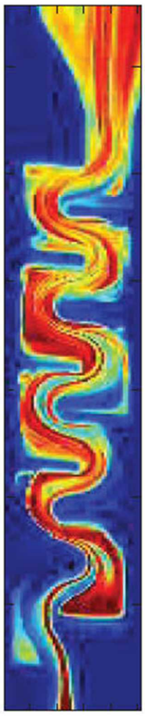
Part II: *Prof. M.A.M. Gijs* (EPFL-LMIS2)

1. Electrohydrodynamics
   Debye-layer, electro-osmotic flow, (di-)electrophoresis
2. Magnetophoresis
3. Nanofluidics

Part I (*T. Lehnert)*

"Microfluidics" -- Thomas Lehnert -- EPFL (Lausanne)

1. INTRODUCTION



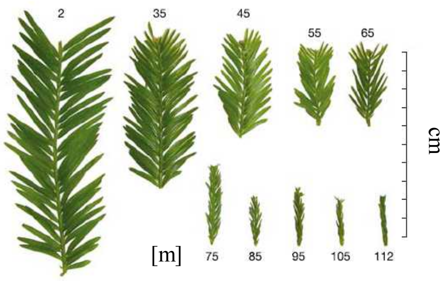
"Microfluidics" -- Thomas Lehnert -- EPFL (Lausanne)

Nature is based on powerful microfluidic systems

Water transport system in vascular plants

Tallest known tree on earth: ***"Hyperion"*** **115.7 m** !
Redwood (*sequoia sempervirens*)

The height seems to be limited by increasing water transport constrains, eventually slowing height growth by reducing photosynthetic carbon gain.



Variation in leaf structure with height in redwood.

G.W. Koch et al., "The limits to tree height"
Nature, Vol 428, p. 851-853 (2004)

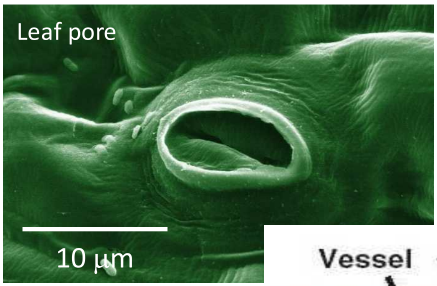
A large oak tree transpires > 400 L/day
[en.wikipedia.org/wiki/Transpiration]

$\Delta p(H_2O)$
10 bar

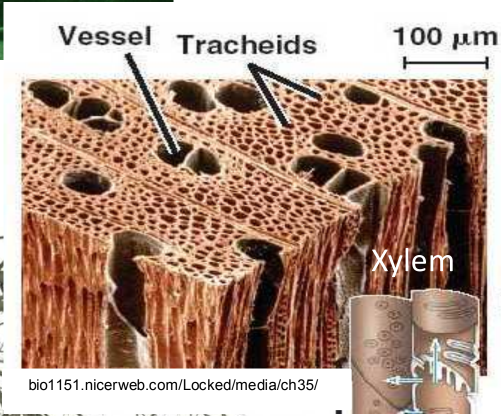
⇨ Microfluidics ?

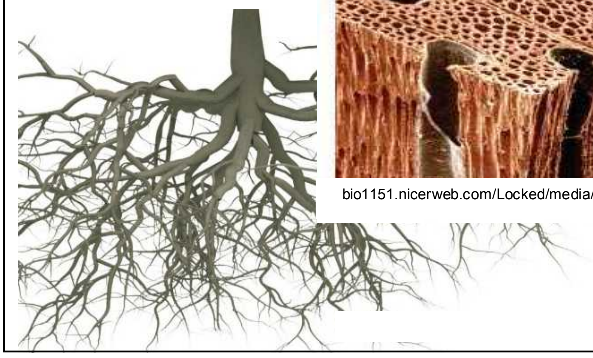
Transport of water occurs through tube-like vessels (∅ ≈ 10 - 100 μm, *Xylem*, ξυλον - wood).

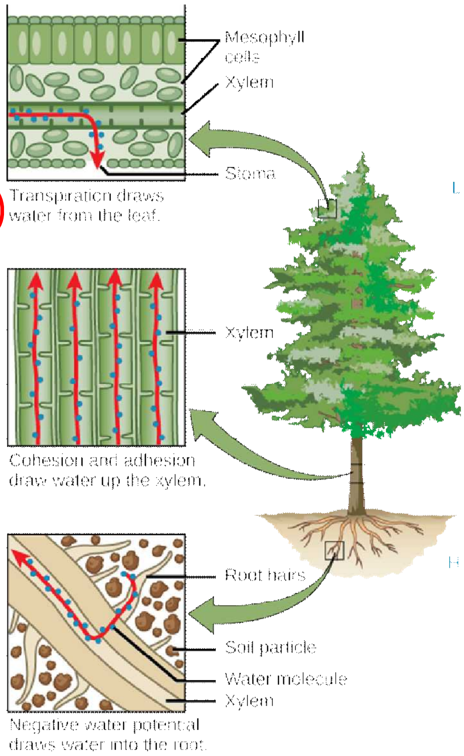
Evaporation through leaf pores (∅ < 10 μm) is one of the major driving forces for pulling up through the tree trunk.



Microfluidics !



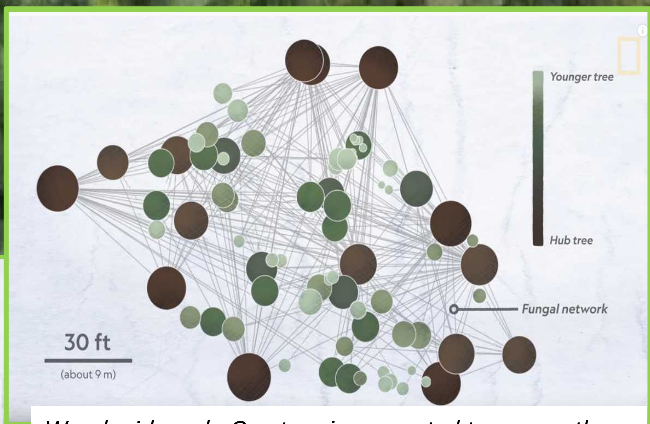




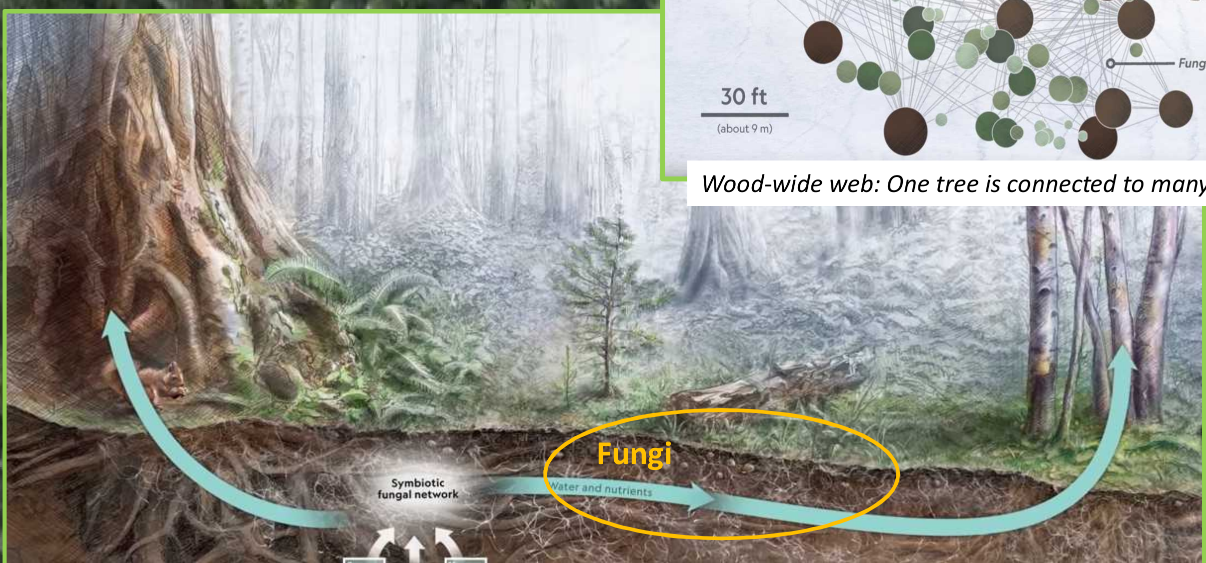
F. Meinzera et al., "Water transport in trees: current perspectives, new insights and some controversies", Env. and Exp. Botany ***45*** *(2001) 239-262*

"***Trees talk to each other***": The whole forest is actually a "microfluidic system" forming an underground communication network of roots and fungal mycelium.

https://www.youtube.com/watch?v=7kHZ0a_6TxY



Wood-wide web: One tree is connected to many others

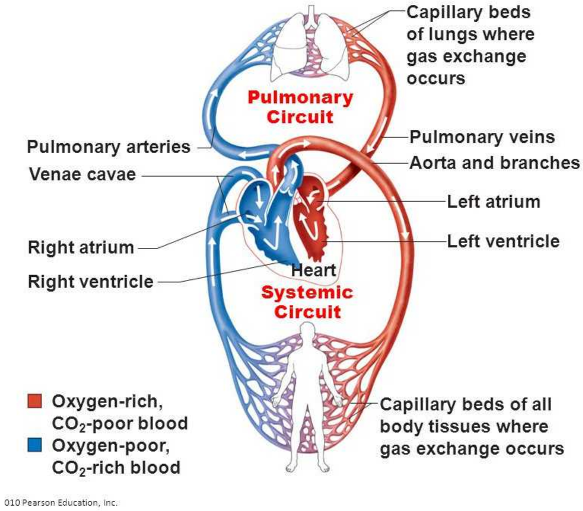


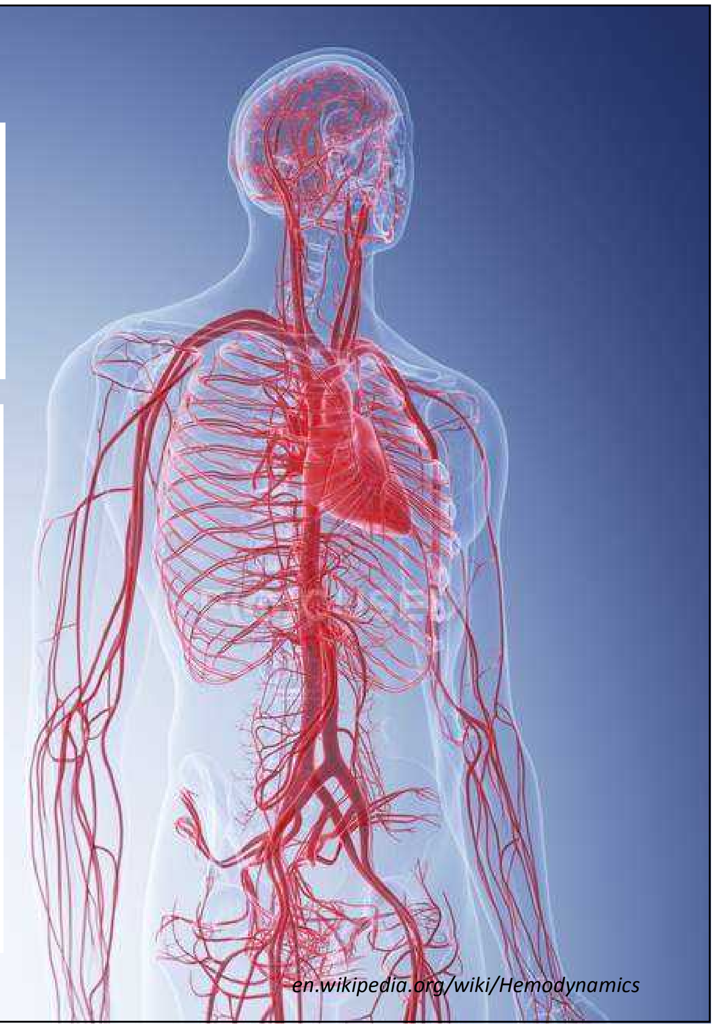
The human cardiovascular system

V ≈ 5L, flow rate at rest ≈ 5L/min.

Networks of blood (and lymphatic) capillaries **(∅ ≈ 5-10 μm)** span over the lung and other organs (exchange of water, O_2/CO_2, nutrients, chemical waste).

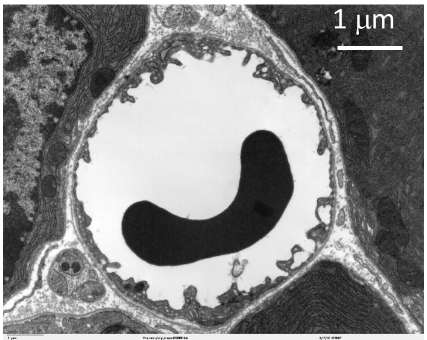
Total length ≈100000 km (!) (80% capillaries).



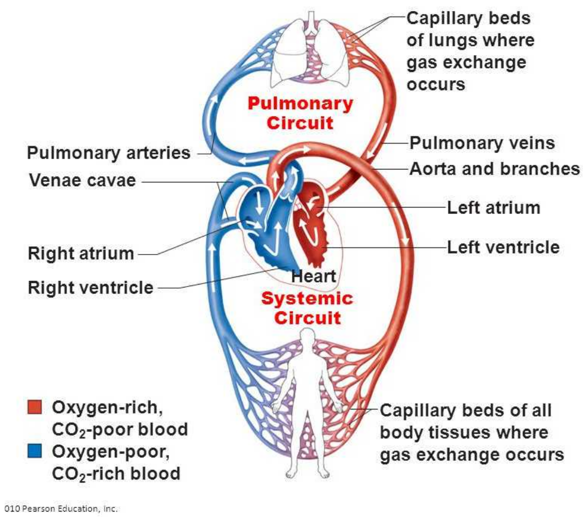


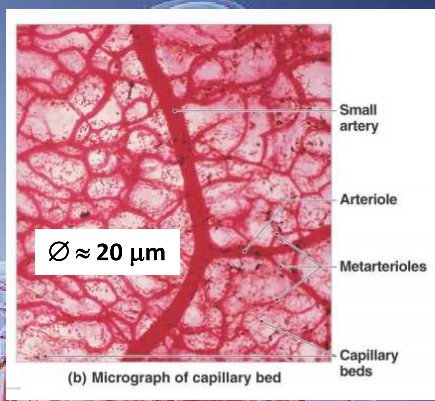
"Microfluidics"

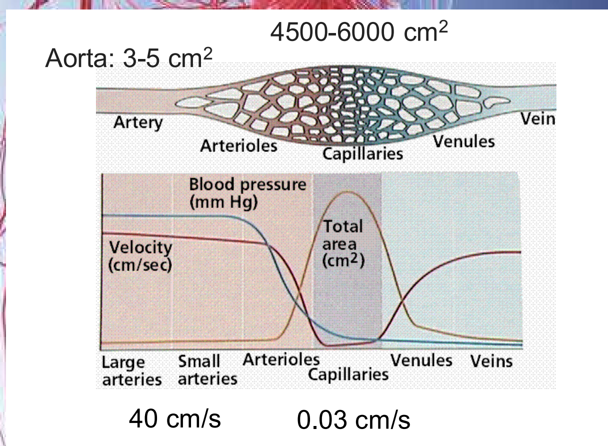
en.wikipedia.org/wiki/Hemodynamics



Red blood cell in a blood capillary



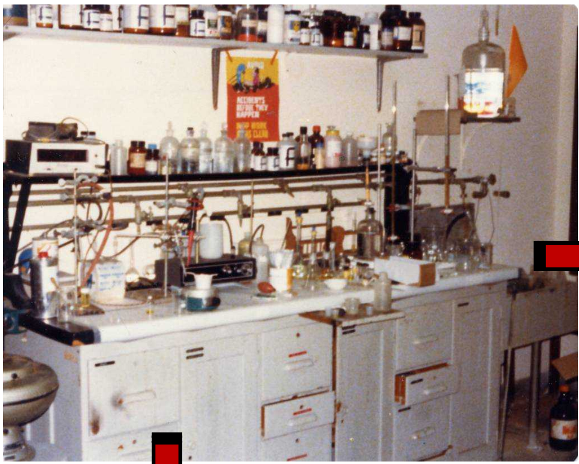




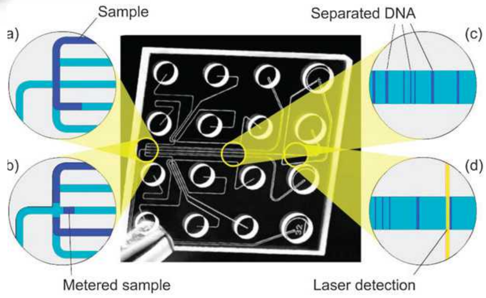
en.wikipedia.org/wiki/Hemodynamics

Microfluidic technologies (Lab-on-a-Chip)

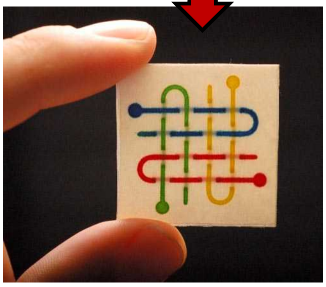
Platforms for bioengineering, bio-chemical assays, bio-analytical applications, etc.



Automated electrophoresis of DNA, RNA, and protein samples (*e.g.* Agilent 2100 Bioanalyzer, Caliper 1999 ⇨)



www.youtube.com/watch?v=wPLzI4bEVc4



Point-of-care testing to perform diagnostic tests without no laboratory support (*e.g.* immunoassays, nucleic acid assays...)

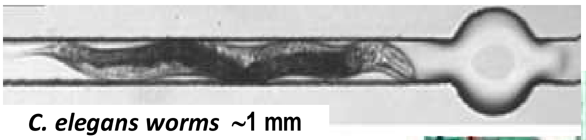
Many review article are available !

P. Abgrall and A-M Gué, Lab-on-chip technologies: making a microfluidic network and coupling it into a complete microsystem—a review, J. Micromech. Microeng. ***17*** *R15, 2007.*

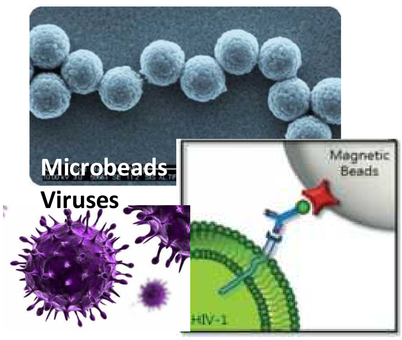
N. Azizipour et al., Evolution of Biochip Technology: A Review from Lab-on-a-Chip to Organ-on-a-Chip, Micromachines, 11, 599, 2020.

Microfluidic and nanofluidic systems

http://htwins.net/scale2/

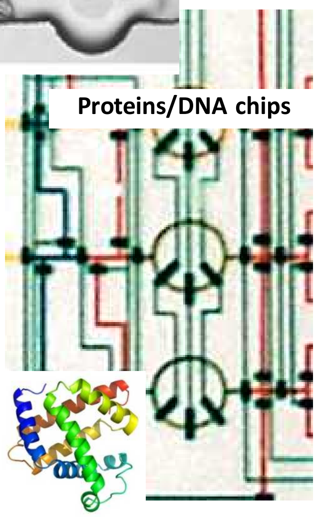


***C. elegans* worms ~1 mm**



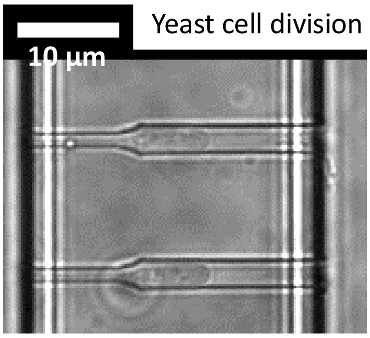
Microbeads

Viruses

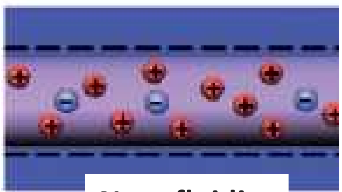


Proteins/DNA chips

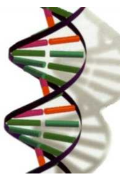
Cells on-chips



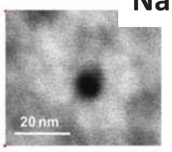
Yeast cell division



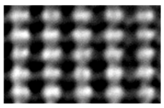
Nanofluidics



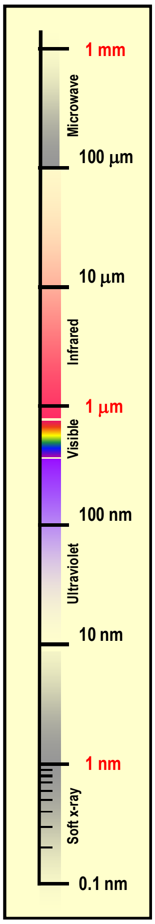
DNA ∅ ~2 nm



Nanopores



Silicon atoms



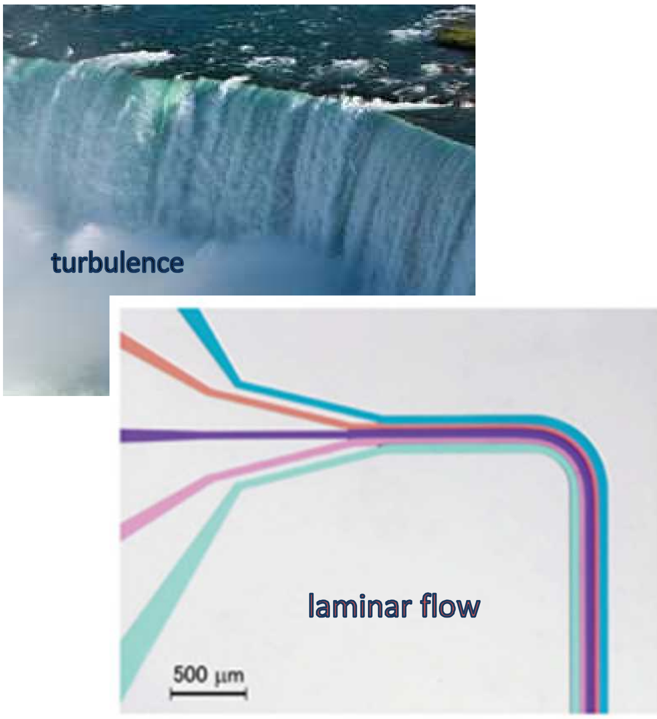
Typical parameter range for microfluidic chips

- Channel width ≤ **100 µm** / height ≈ **1-10 µm**
- Flow rates from **nL/s** to **µL/s**
- Flow velocity **µm/s** to **mm/s** (up to m/s)
- Volumes **µL** to **pL** (*e.g.* droplets)
- Materials: Polymers and glass

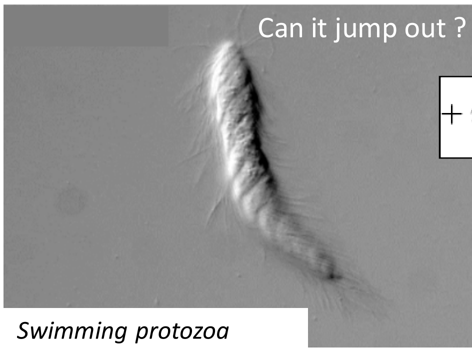
Macrofluidics - Microfluidics: What makes the difference ?

Flow patterns and solid body/liquid interactions

Niagara falls: Average flow rate 2000 m^3/s



Jumping humpback whale
$L \approx$ 15 m, speed 20-50 km/h. Large fins



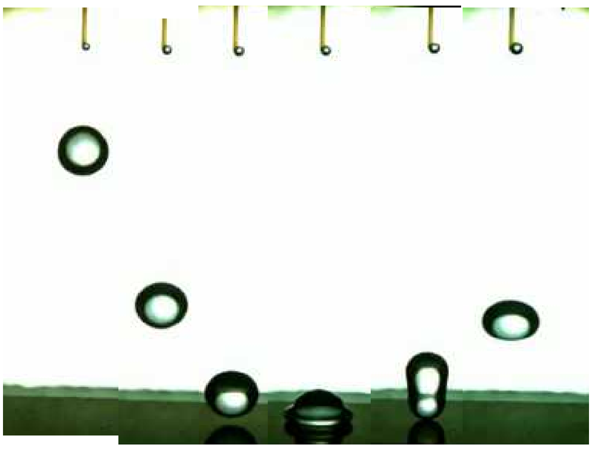
$$+\,\eta\nabla^2\mathbf{v}$$

Swimming protozoa

$L \approx$ 100 µm, speed $\leq$ 100 µm/s - Flagella, cilia

Macrofluidics - Microfluidics: What makes the difference ?

⇨ On the microscale our intuition may fail !



Small bouncing water droplet falling onto a super-hydrophobic surface.

http://www.youtube.com/watch?v=riXp_Q-fDv8

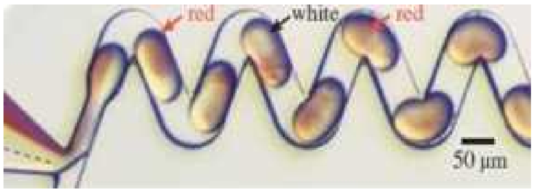
Macrofluidics - Microfluidics: What makes the difference ?

Size matters: Effects of downscaling are important !

$$\frac{surface\ force}{volume\ force} \propto \frac{l^2}{l^3} = l^{-1} \underset{l \to 0}{\longrightarrow} \infty$$

- ⇨ Volume forces (inertial forces and gravity) become negligible. Viscous forces dominate.
- ⇨ Interfacial/capillary forces determine the liquid shape and driving forces. Techniques exploiting boundary effects can be effective in microfluidic systems (*e.g.* electrokinetic effects).
- ⇨ Dimensionless numbers may be defined to evaluate the relative importance of competing forces.

Small raindrop
∅ ≈ mm, $V \approx 50$ µL



Reminder: Some basic mathematical notations

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- Position vector $\mathbf{r} = r_x\,\mathbf{e}_x + r_y\,\mathbf{e}_y + r_z\,\mathbf{e}_z = x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z$ (1.9)
- Einstein summation convention or index notation: Repeated index implies summation over that index.

$$\mathbf{v} = \sum_{i=x,y,z} v_i\,\mathbf{e}_i \equiv v_i\,\mathbf{e}_i \quad \text{vector (e.g. flow velocity)} \tag{1.10}$$

$$\mathbf{v}\cdot\mathbf{u} = v_i u_i \quad \text{scalar or dot product} \tag{1.11}$$

- Differential operators (containing partial spatial derivatives)

$$\boldsymbol{\nabla} \equiv \mathbf{e}_x\partial_x + \mathbf{e}_y\partial_y + \mathbf{e}_z\partial_z = \mathbf{e}_i\partial_i \quad \text{(Nabla operator)} \tag{1.17}$$

$$\Delta \equiv \boldsymbol{\nabla}^2 = \nabla^2 \equiv \partial_i\partial_i \quad \text{(Laplace operator)} \tag{1.18}$$

- The gradient of a scalar field is a vector field $\nabla p = e_x\partial_x p + e_y\partial_y p + e_z\partial_z p = e_i\partial_i p$
- The divergence of a vector field is a scalar field $\nabla\cdot\mathbf{v} \equiv \partial_x v_x + \partial_y v_y + \partial_z v_z = \partial_i v_i$
- The gradient of a vector field is a dyadic product of two vectors (matrix) $\nabla\mathbf{v} \equiv \partial_i v_j$

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- Derivatives of a function $F(\mathbf{r}(t),t)$

  - Partial derivatives $\partial_x F \equiv \dfrac{\partial F}{\partial x}, \quad \text{and} \quad \partial_t F \equiv \dfrac{\partial F}{\partial t},$ (1.15)

  - Total time derivative $d_t F \equiv \dfrac{\mathrm{d}F}{\mathrm{d}t} = \partial_t F + (\partial_t r_i)\partial_i F = \partial_t F + v_i\partial_i F.$ (1.16)

$$d_t F\big(\mathrm{r}(t),t\big) = \partial_t F + (\mathbf{v}\cdot\boldsymbol{\nabla})F \tag{1.19}$$

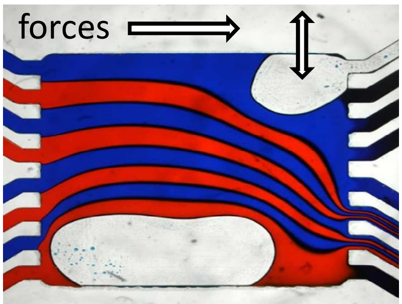
- **Gauss theorem**

  The volume integral over the divergence $\nabla\cdot\mathbf{V}(\mathbf{r})$ of a vector field $\mathbf{V}(\mathbf{r})$ in a region Ω is equal to the surface integral over $\partial\Omega$ of ($\mathbf{n}\cdot\mathbf{V}\,da$), *i.e.* the flux of the quantity $\mathbf{V}(\mathbf{r})$ through a surface area da with the normal vector $\mathbf{n}$.

$$\int_\Omega \mathrm{d}\mathbf{r}\,\boldsymbol{\nabla}\cdot\mathbf{V} = \int_{\partial\Omega} \mathrm{d}a\,\mathbf{n}\cdot\mathbf{V} \quad \text{or} \quad \int_\Omega \mathrm{d}\mathbf{r}\,\partial_j V_j = \int_{\partial\Omega} \mathrm{d}a\,n_j V_j \tag{2.1}$$

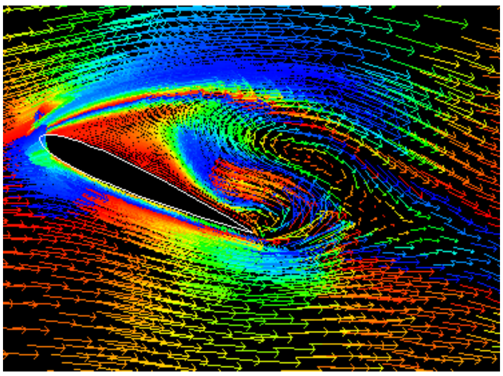
2. Governing equations and flow solutions

2.1 Flow kinematics and shear stress



Microfluidic laminar flow patterns with bubbles

- *Fluid*: Deforms continuously at a measurable rate: **strain rate ε**
- Force gradients or non-uniform stress (force per unit area) change the shape of "fluidic elements".
- Velocity gradients **∇v** are important.
- **Viscous stress** dominates in microfluidic systems !



Continuum description: Fields of macroscopic parameters. Partial differential equations define local properties of a system.

Example: Flow around a wing

v(**r**,*t*) field (arrows) – Vector field: direction and length

p(**r**,*t*) field (colors) – Scalar field: "value", no direction

2.1 Flow kinematics and shear stress

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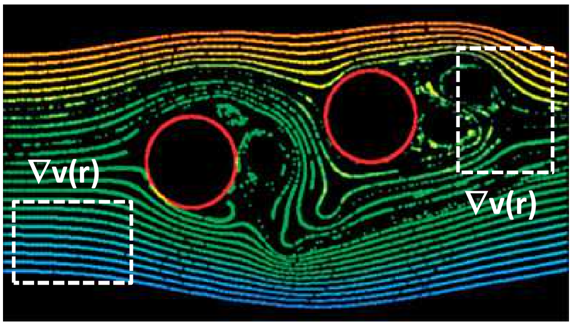
3D velocity (vector) field

$\mathbf{v}(\mathbf{r},t) = \mathbf{v}(v_x,v_y,v_z,t) \equiv \mathbf{u}(\mathbf{r},t) = \mathbf{u}(u,v,w,t)$

(3 spatial components) (+ time)



Streamlines around cylindrical obstacles.

Velocity gradient tensor

(9 components)

General case including all possible velocity field gradients of a velocity field **u**(*u*,*v*,*w*).

$$\nabla\vec{u} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

∇u is a dyadic product of the two vectors ∇ and **u** (rank-2 tensor, **not** the dot product !)

here denotations are taken from: **J. Kirby, *Micro- and nanoscale fluid mechanics : transport in microfluidic devices***

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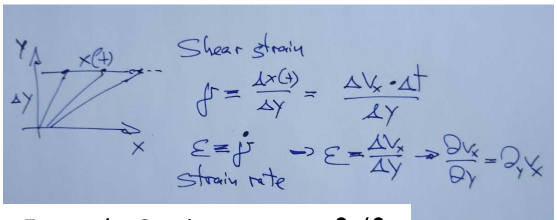
⇨ **Translation:** without deformation (no velocity gradients)

⇨ **Extension/Shear deformation: Symmetric strain rate tensor ε**
(6 components)

The strain rate ε_{ij} [s^{-1}] is a measure for the velocity gradient **∇u** at a given point (**r**,*t*) of the fluidic system.



Example: Strain rate $\varepsilon_{xy} = \partial\gamma/\partial t$ for $\mathbf{v} = v_x(y)$ in *x*-direction.

u ≡ **v**

$$\vec{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z} \end{bmatrix}$$

⇨ **Vorticity**: Fluidic elements may also rotate (not considered here)

here denotations are taken from: **J. Kirby, *Micro- and nanoscale fluid mechanics : Transport in microfluidic devices***

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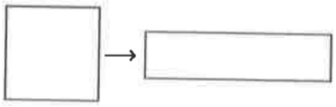
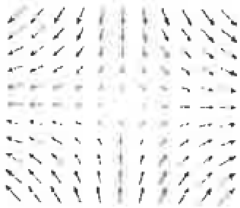
- **Examples: Strain of a fluidic element in specific flow fields**

2D flow fields u(*u*,*v*)

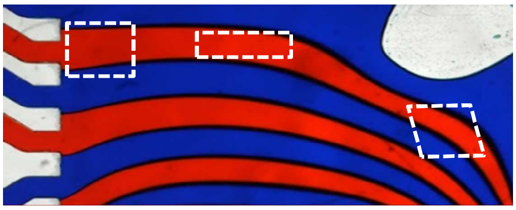


pure extensional strain
(diagonal elements of ε)
Σ = 0 for incompressible fluids !

u = x, v = -y

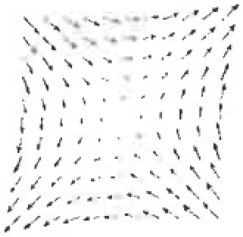


$$\boldsymbol{\varepsilon}_{ext} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

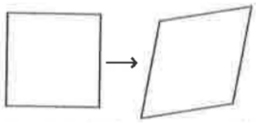


$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} \end{bmatrix}$$

u = y, v = x



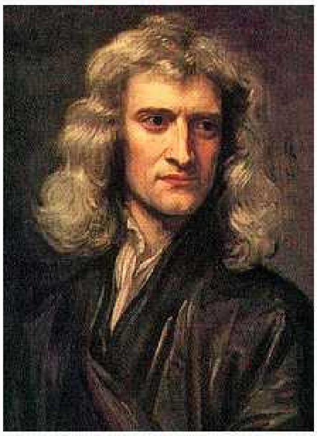
pure shear strain
(symmetric off-diagonal elements of ε)



$$\boldsymbol{\varepsilon}_{shear} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

EPFL

Sir Isaac Newton



Portrait of Isaac Newton in 1689 (age 46) by Godfrey Kneller

Born	25 December 1642 [NS: 4 January 1643][1] Woolsthorpe, Lincolnshire, England
Died	20 March 1726/7 (aged 84) [OS: 20 March 1726 NS: 31 March 1727][1] Kensington, Middlesex, England, Great Britain
Resting place	Westminster Abbey
Residence	England
Nationality	English (later British)
Fields	Physics · Natural philosophy Mathematics · Astronomy Alchemy · Christian theology Economics

⇨ **The strain rate tensor ε is related to the stress tensor σ**

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

$$\boldsymbol{\sigma} = \eta\, \boldsymbol{\varepsilon}$$

NEWTON'S LAW OF VISCOSITY

η (T) [Pa·s] is the dynamic viscosity

η = constant (for T = const) ⇨ **Newtonian fluids**

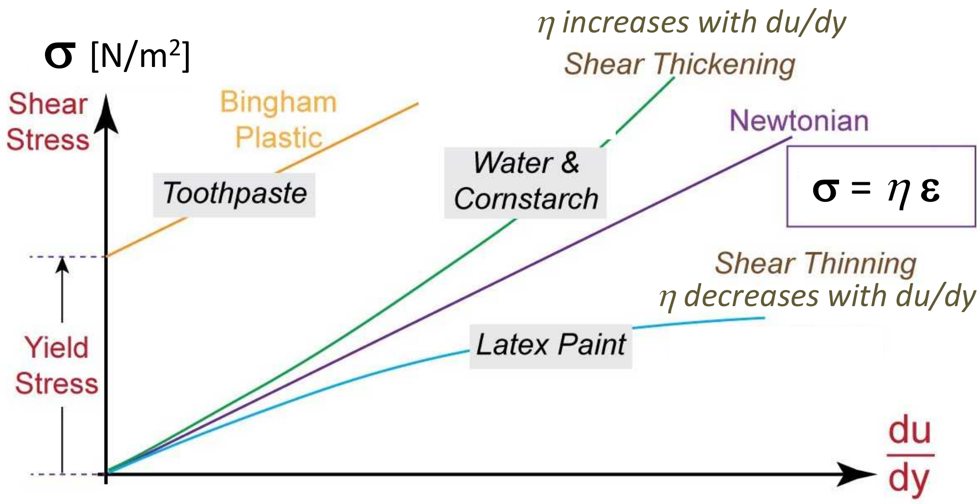
$\eta \rightarrow \eta\,(\nabla u)$ ⇨ Non-Newtonian fluids

In microfluidics the viscous flow regime is predominate, thus the viscous stress tensor is of fundamental importance!

$\boldsymbol{\sigma} \equiv \tau$ in the following sections and in literature !

Non-Newtonian fluids $\eta \rightarrow \eta\,(\nabla u)$

e.g. fluids containing long polymers or colloidal systems (*e.g.* blood).



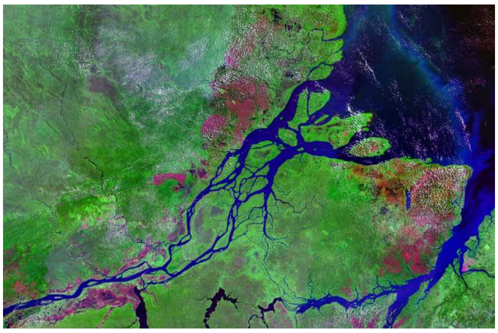
Substance	η in mPa·s
Water (20 °C)	1,00
Blood (37 °C) *Blood is non-Newtonian !*	3 to 25
Honey	$\approx 10^4$

http://en.wikiversity.org/wiki/Fluid_Mechanics_for_MAP/Introduction

Corn starch
https://www.youtube.com/watch?v=v581Y50-bow

Silly Putty
https://www.youtube.com/watch?v=GxdfoJoWNE4

What do we need to determine a flow field **v**(**r**,*t*) ?



- ***Governing equations***
  - ⇨ Continuity equation (mass conservation)
  - ⇨ Navier-Stokes equations (momentum conservation)
- ***Constitutive relations***

  Approximates the response of a material to external stimuli (*e.g.* applied fields or forces). Link between the microscopic properties of the liquid and the macroscopic state (ρ, p, T…), *e.g.* Fourier’s law, Fick’s law etc.

  For viscous flow: $\sigma = \eta\,\varepsilon$
- ***Boundary conditions***

2.2 Continuity equation in fluid dynamics

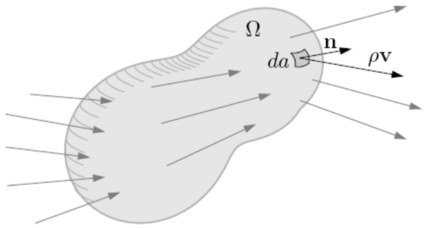
EPFL

- The continuity eqn expresses the conservation of mass $M(\Omega,t)$

$$M(\Omega,t) = \int_\Omega \mathrm{d}\mathbf{r}\, \rho(\mathbf{r},t) \qquad (2.2)$$

$$\mathbf{J}(\mathbf{r},t) = \rho(\mathbf{r},t)\, \mathbf{v}(\mathbf{r},t) \qquad (2.3)$$

with mass flux density $\mathbf{J}(\mathbf{r},t)$ [kg/(m²s)]
mass density ρ, flow velocity $\mathbf{v}$



⇨ $M(\Omega,t)$ in a region Ω can only vary by flow through the surface δΩ

$$\partial_t M(\Omega,t) = \partial_t \int_\Omega \mathrm{d}\mathbf{r}\, \rho(\mathbf{r},t) = \int_\Omega \mathrm{d}\mathbf{r}\, \partial_t \rho(\mathbf{r},t) \qquad (2.4)$$

or

$$\partial_t M(\Omega,t) = -\int_{\partial\Omega} \mathrm{d}a\, \mathbf{n}\cdot\Big(\rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)\Big) = -\int_\Omega \mathrm{d}\mathbf{r}\, \nabla\cdot\Big(\rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)\Big) \qquad (2.5)$$

with (2.4) = (2.5)
and the Gauss theorem (2.1)

$$\boxed{\int_\Omega \mathrm{d}\mathbf{r}\left[\partial_t \rho(\mathbf{r},t) + \nabla\cdot\Big(\rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)\Big)\right] = 0} \qquad (2.6)$$

EPFL

It describes the mass balance in any point of the 3D flow field.

- for compressible fluids with $\rho(\mathbf{r},t)$ and a flow field $\mathbf{v}(\mathbf{r},t)$

$$\boxed{\partial_t \rho + \nabla\cdot(\rho\mathbf{v}) = 0 \quad \text{or} \quad \partial_t \rho + \nabla\cdot\mathbf{J} = 0} \qquad (2.7)$$

$$\nabla \equiv \mathbf{e}_x\partial_x + \mathbf{e}_y\partial_y + \mathbf{e}_z\partial_z = \mathbf{e}_i\partial_i \qquad \boxed{\partial_t \rho = -\partial_j(\rho v_j)} \qquad (2.8)$$

- for incompressible fluids
  (set ρ = const and uniform, *i.e.* $\partial_t\rho = 0$ and $\partial_i\rho = 0$) (2.9)

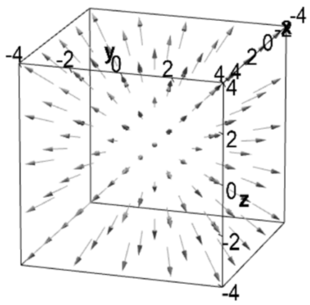
Divergence of $\mathbf{v}(\mathbf{r},t)$

$$\boxed{\nabla\cdot\mathbf{v} = 0 \quad \text{or} \quad \partial_i v_i = 0}$$

EPFL

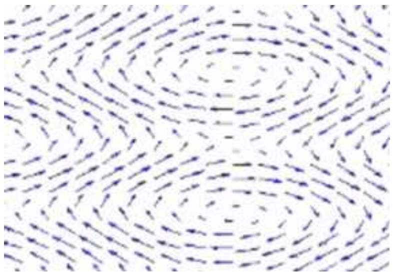
Examples: A physical flow field must fulfill the continuity equation. For incompressible fluids the divergence of $\mathbf{v}(\mathbf{r},t)$ is zero everywhere in the field (no source, no sink).

$$\nabla \cdot \mathbf{v} = 0 \quad \text{or} \quad \partial_i v_i = 0$$



$$\mathbf{v}(x,y,z) = (x, y, z)$$

$$\text{div } \mathbf{v}(x,y,z) = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 1+1+1=3$$



$$\mathbf{v}(x,y,z) = \begin{pmatrix} \sin k_x y \\ \cos k_y x \\ 0 \end{pmatrix}$$

div **v** = 0

EPFL

2.3 Navier-Stokes equations

Claude-Louis Navier

Bust of Claude Louis Marie Henri Navier at the École Nationale des Ponts et Chaussées

Born	10 February 1785 Dijon, France
Died	21 August 1836 (aged 51) Paris, France
Nationality	French

A specialist in bridge building (he was the first to develop a theory of suspension bridges).

1821/1822

Navier modified the *Euler equations* for inviscid flow [1,2]

$$\rho\left(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + \qquad + \rho\,\mathbf{g}$$

…by introducing the viscous term

$$+\eta\nabla^2\mathbf{v}$$

Euler incorrectly assumed that, similar to the case of friction in solids, fluid friction was proportional to pressure.

"The irony is that although Navier had no conception of shear stress, he nevertheless arrived at the proper form for such equations."

J D Anderson, *A History of Aerodynamics* (Cambridge, 1997)

[1] Euler, Leonhard (**1757**). "Principes généraux de l'état d'équilibre d'un fluide". Mémoires de l'académie des sciences de Berlin. 11: 217–273

[2] *Refs in* S. R. Bistafa, "On the development of the Navier–Stokes equation by Navier", http://dx.doi.org/10.1590/1806-9126-rbef-2017-0239

- The Navier-Stokes equations (NSE) are the **equations of motion for fluid flow**.
- The NSE are a set of 3 non-linear 2nd order partial differential equations for the $v_x(t)$, $v_y(t)$, $v_z(t)$ components of a flow field $\mathbf{v}(\mathbf{r},t)$.
- They express the **conservation of linear momentum $\mathbf{P}(\Omega,t)$** in any point of the flow field $\mathbf{v}(\mathbf{r},t)$.
- They describe the transport by advection.
- They may be considered as Newton's 2nd law applied to fluid mechanics.

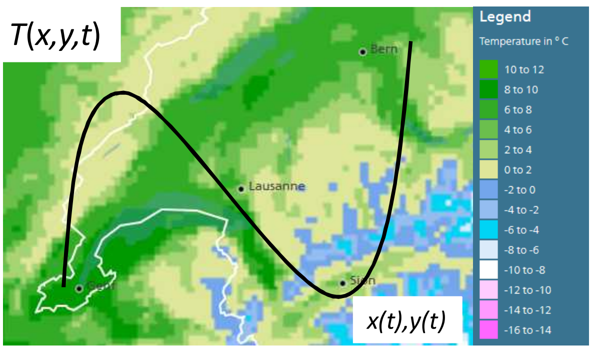
⇨ *In the following the Navier-Stokes eqns will first be derived by using the Lagrange derivative and a heuristic model.*

Navier-Stokes eqns derived by using the Lagrange derivative

EPFL

A particle is moving on a path through a 3D field $\Phi(x,y,z,t)$. The pathline is given by $[x(t),y(t),z(t)]$.

How does Φ change along the path of the particle ?



⇒ Determine the Lagrange derivative for a specific path (also called substantial or material derivative).

⇒ Variations of $\Phi(x(t),y(t),z(t),t)$ along the path may be determined by applying the chain rule for the time derivative.

$\Phi(x(t),y(t),z(t),t)$

$$\frac{\mathrm{D}\Phi}{\mathrm{D}t} \equiv \frac{\partial\Phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial\Phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial\Phi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial\Phi}{\partial t}$$

or with $v_i = \mathrm{d}x_i/\mathrm{d}t$

$$\frac{\mathrm{D}\Phi}{\mathrm{D}t} \equiv \frac{\partial\Phi}{\partial x}v_x + \frac{\partial\Phi}{\partial y}v_y + \frac{\partial\Phi}{\partial z}v_z + \frac{\partial\Phi}{\partial t}$$

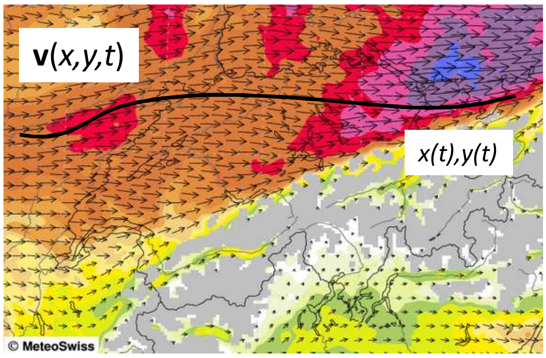
The 3D Lagrange derivative can be written as

$$D_t = \partial_t + (\mathbf{v}\cdot\boldsymbol{\nabla}) \qquad (2.34)$$

Navier-Stokes eqns derived by using the Lagrange derivative

EPFL

How does the velocity **v***(x,y,z,t) of a particle change on a path through a flow field* **v***(***r***,t)?*



In this case trajectory and velocity of the particle are not arbitrary but determined by the flow field **v***(x,y,z,t)* itself.

The Lagrange derivative for the velocity component v_x experienced by the particle can be written as (likewise for v_y and v_z):

$$\frac{\mathrm{D}v_x}{\mathrm{D}t} \equiv \frac{\partial v_x}{\partial x} v_x + \frac{\partial v_x}{\partial y} v_y + \frac{\partial v_x}{\partial z} v_z + \frac{\partial v_x}{\partial t}$$

Using the previous notation $D_t = \partial_t + (\mathbf{v}\cdot\boldsymbol{\nabla})$ (2.34)

and Newton's 2nd law $m\,\mathrm{d}_t\mathbf{v} = \sum_j \mathbf{F}_j$ (2.31)

The equation of motion takes the form of the Navier-Stokes equation

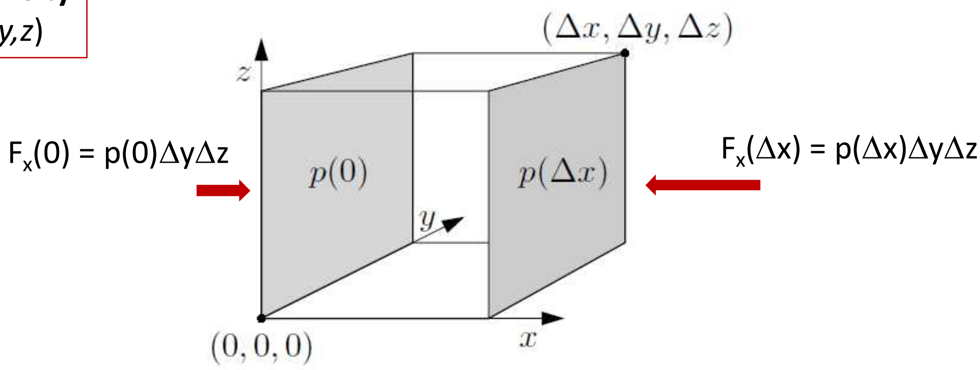
$$\rho\, D_t\mathbf{v} = \sum_j \mathbf{f}_j \quad \text{or} \quad \boxed{\rho\big[\partial_t\mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\big] = \sum_j \mathbf{f}_j} \qquad (2.35)$$

where $\mathbf{f}_j$ are force densities related to pressure, viscosity and external body forces.

A heuristic derivation of the pressure and viscosity force densities

EPFL

Pressure force density
f$(x,y,z) = -\nabla p(x,y,z)$



Total pressure force in *x* $\quad F_x = p(0)\,\Delta y\Delta z - p(\Delta x)\,\Delta y\Delta z$

Total force density in *x*

$$f_x = \frac{p(0)\,\Delta y\Delta z - p(\Delta x)\,\Delta y\Delta z}{\Delta x\Delta y\Delta z} = -\frac{p(\Delta x) - p(0)}{\Delta x} \underset{\Delta x\to 0}{\longrightarrow} -\partial_x p, \qquad (2.74)$$

which is the x component of $\mathbf{f} = -\boldsymbol{\nabla} p$. The other two components are derived similarly.

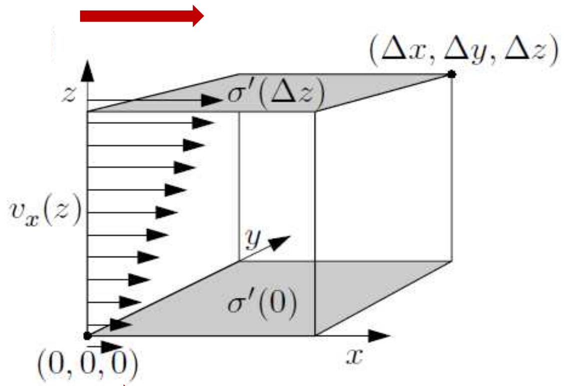
A heuristic derivation of the pressure and viscosity force densities

Viscous force density
$\mathbf{f}(x,y,z) = \eta \nabla^2 \mathbf{v}(x,y,z)$

$$\boldsymbol{\sigma} = \eta\, \boldsymbol{\varepsilon}$$

Stress/strain rate relation-ship for Newtonian fluids

$F_x(\Delta z) = \eta\, \partial_z v_x(\Delta z)\, \Delta x \Delta y$



$F_x(0) = \eta\, \partial_z v_x(0)\, \Delta x \Delta y$

$$f_x = \eta \frac{\partial_z v_x(\Delta z)\, \Delta x \Delta y - \partial_z v_x(0)\, \Delta x \Delta y}{\Delta x \Delta y \Delta z} = \eta \frac{\partial_z v_x(\Delta z) - \partial_z v_x(0)}{\Delta z} \xrightarrow[\Delta z \to 0]{} \eta \partial_z^2 v_x, \quad (2.75)$$

The Navier-Stokes equations for incompressible fluids

Advective terms account for **acceleration** of fluidic particles in unsteady or steady flow states.

⇨ **Inertial force densities**

$$\rho\Big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\Big) = -\boldsymbol{\nabla} p + \eta \nabla^2 \mathbf{v} + \rho\, \mathbf{g} + \rho_{\mathrm{el}} \mathbf{E}$$

The transient term $\partial_t \mathbf{v}$ is relevant if $\mathbf{v}(t)$ changes with time.

The non-linear term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ describes convective acceleration (time-independent), *e.g.* in systems with no translation invariance.

$(\mathbf{v}\cdot\nabla)\mathbf{v}$ is particularly relevant in turbulent flow regimes.

The Navier-Stokes equations for incompressible fluids

EPFL

Body force densities

$$\rho\Big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\Big) = -\boldsymbol{\nabla} p + \eta\nabla^2\mathbf{v} + \rho\,\mathbf{g} + \rho_{\mathrm{el}}\mathbf{E}$$

∇p and $\nabla\cdot\boldsymbol{\sigma}$ are surface force densities for generated by pressure and viscous shear stress.

For incompressible fluids $\nabla\cdot\boldsymbol{\sigma} = \eta\nabla^2\mathbf{v}$

η dynamic viscosity [Pa·s]

Navier-Stokes equations in Cartesian coordinates

EPFL

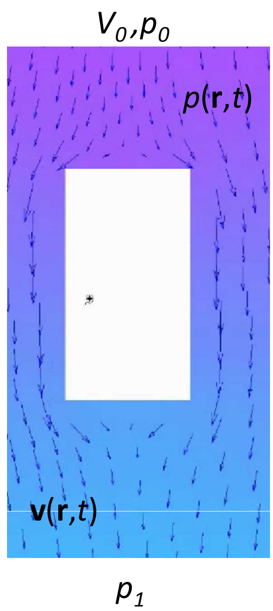
3 eqns and the continuity eqn are required to determine the 4 unknown quantities v_x, v_y, v_z and p. These equations describe the velocity and pressure fields in each point (x,y,z) of a fluidic system (ρ, η = const).

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \eta\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \eta\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \eta\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right)$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$



Boundary conditions (*e.g.* defined by the geometry of the device, the pressure at inlet/outlet, fluidic interfaces, etc) determine the actual flow patterns.

-> Analytical solutions can be found only in specific cases.

Exploring the non-linear term $(v\cdot\nabla)v$ in the Navier-Stokes eqns

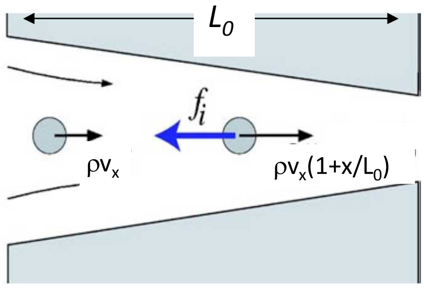
EPFL

Example: Steady (time independent) flow through a constriction (nozzle)

⇨ Advective acceleration of the flow

⇨ The inertial part of the NSE is given by

$$\rho\Big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\Big) \quad \textit{with} \quad \partial_t \mathbf{v} = 0$$



Rough estimate of the inertial force density $\boldsymbol{f}_i$, assuming that v_x increases by V_0 over L_0.
V_0 and L_0 are characteristic scales of the system.

$$f_i = \rho V_0 \frac{dv_x}{dx} \sim \frac{\rho V_0^2}{L_0} \qquad [\,\mathrm{N \cdot m^{-3}}\,]$$

T. M. Squires and S. R. Quake: Microfluidics: Fluid physics at the nanoliter scale

Exploring the non-linear term $(v\cdot\nabla)v$ in the Navier-Stokes eqns

EPFL

Example: Inviscid flow (neglecting viscosity) in a pressure field.

$$(\mathbf{v}\cdot\nabla)\mathbf{v}\Big) = -\nabla p$$

e.g. only pressure gradient in *x*-direction

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x}$$

A pressure force in *x*-direction generates velocity/gradient components in *x,y,z* directions. This results in flow instabilities and turbulences !

Turbulences on marcoscale due to fluidic inertia

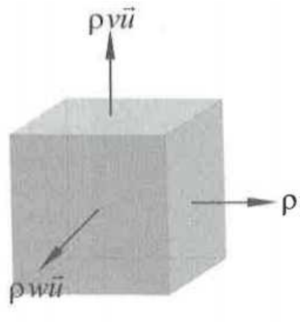
In microfluidics inertial forces are normally negligible with respect to viscous forces (see below the discussions on Reynolds number and Stokes flow).

Examples of microfluidic systems where inertial fluidic properties are relevant will be shown later (*e.g.* secondary Dean flow for mixing).

Navier-Stokes equations (General theoretical derivation)

More details in *Henrik Bruus* "Theoretical Microfluidics"

$P(\Omega,t)$ in a region Ω can vary by **momentum flow** (convection) through the surface $\delta\Omega$ and by **action of forces**.

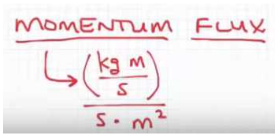


$$\partial_t P_i^{\mathrm{conv}}(\Omega,t) = -\int_{\partial\Omega} \mathrm{d}a\, \mathbf{n}\cdot(\rho v_i\, \mathbf{v}) \qquad (2.13)$$

Momentum flux density tensor $\mathbf{\Pi}$

$$\Pi' \equiv \rho\mathbf{v}\mathbf{v} \qquad \Pi'_{ij} \equiv \rho v_i v_j \qquad (2.12)$$

The total rate of change of the i-th component $\partial_t P_i$ of the momentum in the region Ω is given by the surface integral $\partial\Omega$ over the flow $\mathbf{n}\cdot(\rho v_i)\mathbf{v}$ da of the i-th component through (da $\mathbf{n}$).



$$\partial_t P_i(\Omega,t) = \partial_t P_i^{\mathrm{conv}}(\Omega,t) + \partial_t P_i^{\mathrm{pres}}(\Omega,t) + \partial_t P_i^{\mathrm{visc}}(\Omega,t) + \partial_t P_i^{\mathrm{body}}(\Omega,t) \qquad (2.11)$$

- **Viscous forces on $\partial\Omega$**

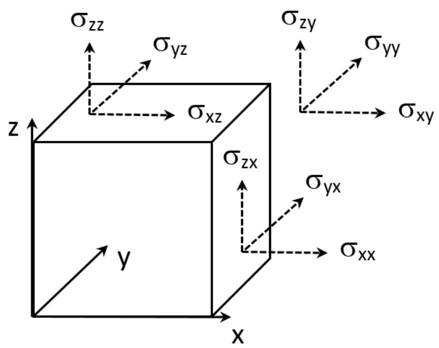
The viscous **stress tensor σ'** relates all possible combinations of surface force components and surface orientations. The stress tensor σ allows calculating stress forces occurring on any arbitrary surface of a control volume.

$$\partial_t P_i^{\mathrm{visc}}(\Omega,t) = \int_{\partial\Omega} \mathrm{d}a\, n_j\, \sigma'_{ij} \qquad (2.16)$$

$$\sigma'_{ij} = \eta\left(\partial_i v_j + \partial_j v_i\right) \qquad (2.17)$$

$$\mathrm{d}F_i = \sigma'_{ij} n_j\, \mathrm{d}a$$

i-th component of the friction force acting on a surface element (n_j da)



$$\sigma'_{xx} = \eta\,(\partial_x v_x + \partial_x v_x) = 0 \qquad (\rho = \text{const, no } p)$$

$$\sigma'_{xy} = \eta\,(\partial_x v_y + \partial_y v_x) \qquad \mathrm{d}F_x = \sigma'_{xy}\, n_y\, \mathrm{d}a$$

$$\sigma'_{xz} = \eta\,(\partial_x v_z + \partial_z v_x) \qquad \mathrm{d}F_x = \sigma'_{xz}\, n_z\, \mathrm{d}a$$

Remark: In other literature the viscous stress tensor is often denoted τ ($\tau \equiv \sigma$). The indices τ_{ij} also are often inversed, *i.e.* the first index indicates the surface normal, and the second index the force direction.

EPFL

- **Viscous forces on $\partial\Omega$**

$$\partial_t P_i^{\mathrm{visc}}(\Omega,t) = \int_{\partial\Omega} \mathrm{d}a\, n_j\, \sigma'_{ij} \qquad (2.16)$$

Compressible fluid

$$\sigma'_{ij} = \eta\left(\partial_j v_i + \partial_i v_j\right) + \left(\zeta - \tfrac{2}{3}\eta\right)\left(\partial_k v_k\right)\delta_{ij} \qquad (2.18b)$$

Incompressible fluid (uniform viscosity η)

$$\sigma'_{ij} = \eta\left(\partial_i v_j + \partial_j v_i\right) \quad \text{and} \quad \partial_k v_k = 0 \qquad (2.20)$$

The shear stress tensor $\boldsymbol{\sigma}'$ is symmetric $\sigma'_{ij} = \sigma'_{ji}$
Shear forces appear as off-diagonal elements.

η is the dynamic viscosity due to shear stress.
ζ stands for internal friction due to compression/stretching.

- **Pressure forces on $\partial\Omega$**

Normal to the surface, may be included in $\boldsymbol{\sigma}$ as diagonal elements

$$\sigma_{ij} \equiv -p\,\delta_{ij} + \sigma'_{ij} \qquad (2.26)$$

EPFL

$\Rightarrow$ *Applying the Gauss theorem to the net rate of change $\partial_t P_i$ in Ω in eqn. (2.11)*

- Navier-Stokes equation for compressible Newtonian fluids

η = const, ζ = const, and $\rho(\mathbf{r},t)$

$$\rho\Big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\Big) = -\boldsymbol{\nabla}p + \eta\nabla^2\mathbf{v} + \Big(\frac{1}{3}\eta + \zeta\Big)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) + \rho\,\mathbf{g} + \rho_{\mathrm{el}}\mathbf{E} \qquad (2.29)$$

η is the dynamic viscosity due to shear stress.

ζ stands for internal friction due to compression/extension.

- Navier-Stokes equation for incompressible Newtonian fluids

ρ = const, η = const

$$\rho\Big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\Big) = -\boldsymbol{\nabla}p + \eta\nabla^2\mathbf{v} + \rho\,\mathbf{g} + \rho_{\mathrm{el}}\mathbf{E} \qquad (2.30)$$

2.4 Solutions for simple flow problems

- The US$ 1 million problem: Navier-Stokes equations are a system of non-linear coupled partial differential eqns. $\rho(\mathbf{v}\cdot\nabla)\mathbf{v}$ accounts for interesting hydrodynamic phenomena, but…they are unsolved !

Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations.

CMI Clay Mathematics Institute
Dedicated to increasing and disseminating mathematical knowledge

http://www.claymath.org/millennium-problems

Approaches for solving the Navier-Stokes equations

Analytical techniques (direct integration, eigenfunction expansion etc.) may be used only for simple geometries, otherwise numerical solutions.

Simplifications of the NSE may be possible in specific cases (*e.g.* Stokes eqn at low *Re* number, Laplace eqn for irrotational flow, boundary eqn).

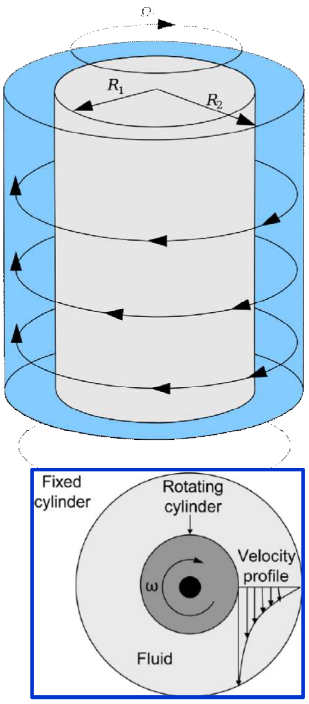
Initial and boundary conditions for **v** and/or **∇v**, stress components, and pressure have to be defined to determine the flow and pressure fields for a given system.

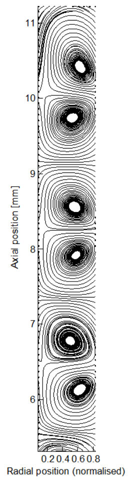
$$\mathbf{v}(\mathbf{r}) = \mathbf{0}, \quad \text{for } \mathbf{r} \in \partial\Omega \ \text{(no-slip)} \tag{3.1}$$

No-slip boundary condition for a motionless wall

Taylor-Couette flow

Viscous fluid confined in the gap between two rotating cylinders.
Good model system to study flow instabilities and transitions.





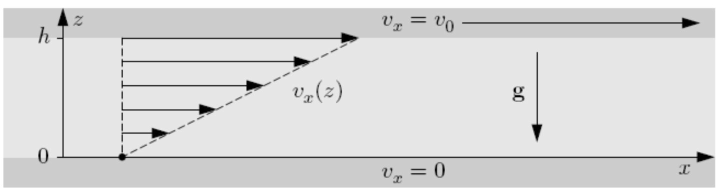
Re < *Re*$_{critical}$
Couette flow purely azimuthal and laminar (bearing flow).
Application: Rheometers for measuring η

Re > *Re*$_{critical}$
Flow becomes unstable and toroidal vortices emerge (**Taylor vortex flow**).

Re >> *Re*$_{critical}$
Flow has different turbulent patterns.

https://doi.org/10.1051/epjconf/201921302014
https://en.wikipedia.org/wiki/Taylor-Couette_flow

Couette flow between two moving parallel plates



(Fig. 3.3)

Fluid between two infinite parallel plates. The top plate moves in x-direction with constant speed v_0, resulting in a linear flow velocity profile $v_x(z)$ in z-direction.

- Navier-Stokes eqn (*p* = const) $\boxed{\eta\, \partial_z^2 \mathbf{v} = 0}$ (3.13)

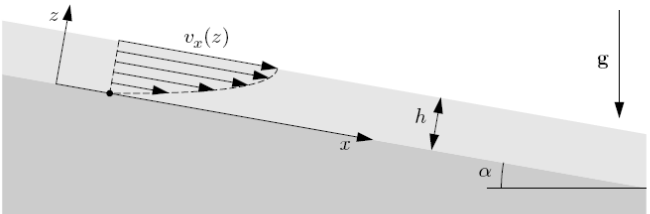
  - *Translation invariance along x, y* $\mathbf{v}(\mathbf{r}) = v_x(z)\, \mathbf{e}_x$ *and* $(\mathbf{v}\cdot\nabla)\mathbf{v} = 0$

  - *using the no-slip boundary conditions* $v_x(0) = 0$ and $v_x(h) = v_0$

- Solution: Linear velocity profile $v_x(z) = v_0\, \frac{z}{h}$ (3.15)

Force required to move the plate (with surface *A*) $F_x = \sigma'_{xz}\, \mathcal{A} = \eta\, \frac{v_0 \mathcal{A}}{h}$ (3.16)

Liquid film flow on an inclined plane



(Fig. 3.2)

Liquid film flowing down the plane has a half-parabolic velocity profile in steady-state.
(e.g. a 100 μm thick water film flows at a mean velocity of 1 cm/s)

- Navier-Stokes eqn

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \eta\,\partial_z^2\mathbf{v} + \rho g \sin\alpha\, \mathbf{e}_x \quad (3.9)$$

with $\mathbf{v}(\mathbf{r}) = v_x(z)\,\mathbf{e}_x$ *and* $(\mathbf{v}\cdot\nabla)\mathbf{v} = v_x(z)\partial_x[v_x(z)] = 0$ (3.10)

⇒ 2^{nd} order linear eqn $\eta\,\partial_z^2 v_x(z) = -\rho g \sin\alpha$ (3.11)

boundary conditions $v_x(0) = 0$ no-slip
$\eta\delta_z v_x(h) = 0$ no stress

- Half-parabola solution

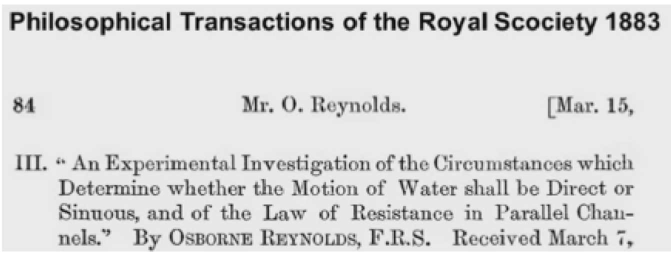
$$v_x(z) = \sin\alpha\,\frac{\rho g}{2\eta}\,(2h - z)z \quad (3.12)$$

⇨ see also the full parabolic profile for *Poiseuille flow* discussed in Chapter 3.

2.5 Reynolds number and Stokes flow

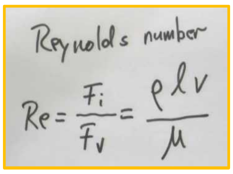
Osborne Reynolds

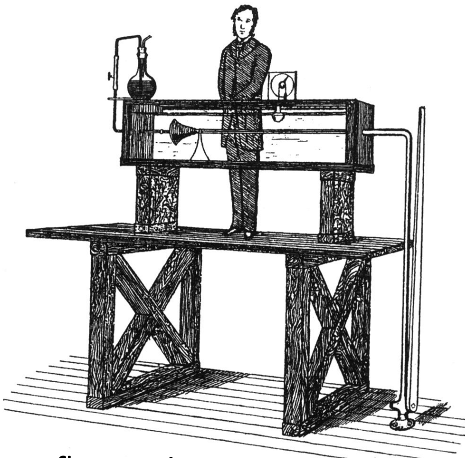
Born	23 August 1842 Belfast, Ireland
Died	21 February 1912 (aged 69) Watchet, Somerset, England
Nationality	United Kingdom
Fields	Physics


Philosophical Transactions of the Royal Scociety 1883

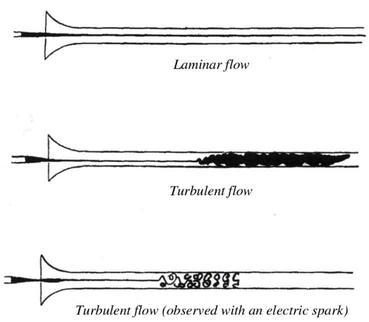
84 Mr. O. Reynolds. [Mar. 15,

III. " An Experimental Investigation of the Circumstances which Determine whether the Motion of Water shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels." By OSBORNE REYNOLDS, F.R.S. Received March 7,





Flow regimes



for $Re \geq Re_{crit} >> 1 \Rightarrow$ turbulent flow regime

Different flow regimes

EPFL

Transition from laminar to turbulent flow in a tube with increasing flow speed

(Reproduction of Reynold's original experiment)

https://www.youtube.com/watch?v=XOLl2KeDiOg

Dimensionless form of the Navier-Stokes equations

EPFL

Navier-Stokes eqn

$$\rho\Big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\Big) = -\boldsymbol{\nabla} p + \eta\nabla^2\mathbf{v}$$

The general fluidic properties of a system can be evaluated by using **characteristic scales** determined by the boundary conditions: L_0 , V_0

Dimensionless (normalized) forms can be derived for all variables, for instance…

$$\mathbf{r} = L_0\,\tilde{\mathbf{r}} \qquad \mathbf{v} = V_0\,\tilde{\mathbf{v}} \qquad \boldsymbol{\nabla} = (1/L_0)\,\tilde{\boldsymbol{\nabla}}$$

In microfluidics, the pressure p is normalized by a characteristic shear stress $\eta V_0/L_0$.

$$p = \frac{\eta V_0}{L_0}\,\tilde{p} = P_0\,\tilde{p} \qquad (2.36)$$

Dimensionless form of the Navier-Stokes equations

➩ Making the Navier-Stokes eqns dimensionless

if $\partial_t \mathbf{v} = 0$

$$\rho\left[\frac{V_0{}^2}{L_0}\,(\tilde{\mathrm{v}}\cdot\tilde{\nabla})\tilde{\mathrm{v}}\right] = -\frac{P_0}{L_0}\,\tilde{\nabla}\tilde{p} + \frac{\eta V_0}{L_0{}^2}\,\tilde{\nabla}^2\tilde{\mathrm{v}} \qquad (2.37)$$

Dimensionless form of the NSE

$$\boxed{Re\left[(\tilde{\mathrm{v}}\cdot\tilde{\nabla})\tilde{\mathrm{v}}\right] = -\tilde{\nabla}\tilde{p} + \tilde{\nabla}^2\tilde{\mathrm{v}}} \qquad (2.38)$$

Reynolds number

$$\boxed{Re = \frac{\rho\, V_0\, L_0}{\eta}} \quad ➩ \quad \frac{\rho V_0{}^2/L_0}{\eta V_0/L_0{}^2} \rightarrow \frac{inertial\ force}{viscous\ force} \qquad (2.39)$$

if $\partial_t \mathbf{v} \neq 0$

$$\rho\left[\frac{Re}{St}\,\tilde{\partial}_t\tilde{\mathrm{v}} + Re\left[(\tilde{\mathrm{v}}\cdot\tilde{\nabla})\tilde{\mathrm{v}}\right]\right] = -\tilde{\nabla}\tilde{p} + \tilde{\nabla}^2\tilde{\mathrm{v}} \qquad (2.37)$$

Strouhal number

$$\boxed{St = T_{ch}V_0/\,L_0}$$

St accounts for a characteristic time scale T_{ch}, *e.g.* defined by oscillating boundaries. Describes internal flow instabilities (vortex formation).

Some values for Reynolds numbers

Re numbers are important for scaling of fluidic systems.

Re numbers may be defined in different ways, depending on the characteristic scales of the specific system (*e.g.* width of a channel, length of a body, etc).

Typical values for microfluidic devices:

water, 1 µm/s - 1 cm/s, channel 1 - 100 µm

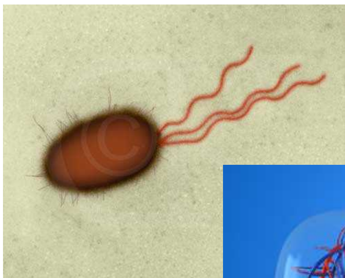
➩ *Re* range between $\mathcal{O}(10^{-6})$ to $\mathcal{O}(10^{0})$

Microorganisms	**~ 10^{-6} - 10^{-3}**
Blood flow in brain	**~ 1×10^2**
Blood flow in aorta	**~ 1×10^3**

Human swimming	~ 10^4 - 10^6
Fastest Fish	~ 10^6
Large ship	~ 10^9

Onset of turbulent flow

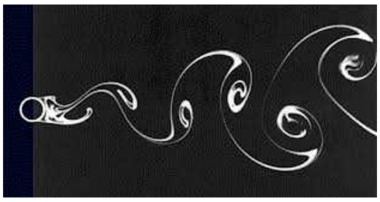
Flow in a pipe	2.3×10^3 to 5.0×10^4
Boundary layers	up to 10^6



Example: Flow past a sphere for rising *Re* numbers

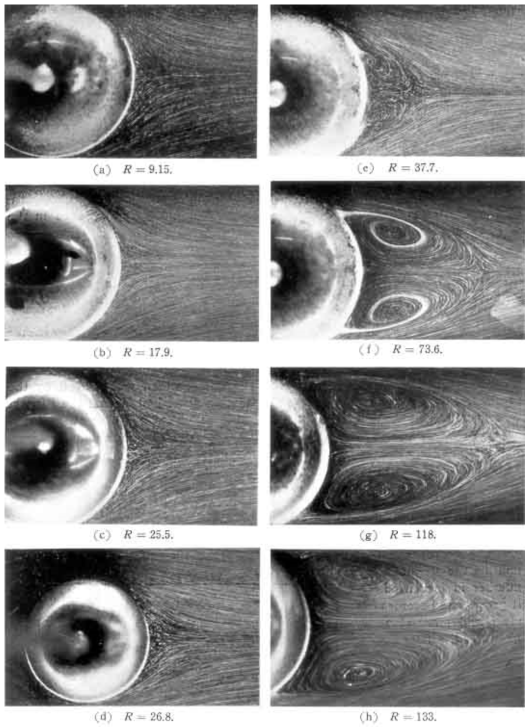
Streamlines of a flow over a sphere with increasing flow speed (water flowing from left to right, ∅ ≈ 19 mm). The inertial **(v·∇)v** term becomes increasingly important.

- Laminar flow for low *Re* < 10
- Vortex ring develops at *Re* ≅ 30
- Wake increases in size, becomes comparable to sphere size at *Re* ≅ 130
- Wake remains attached for *Re* < 500
- *Re* > 500, vortices begin to be shed



Karman vortex street

Patterns for rising Re

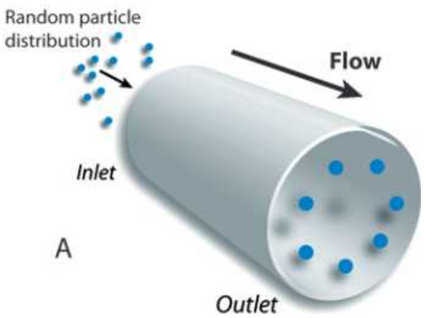


S. Taneda, J. Phys. Soc. Japan, Vol. 11(10), p. 1104-1108 (1956)

Inertial microfluidics

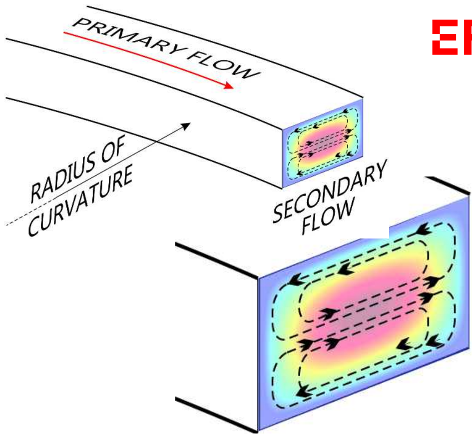
In microfluidics fluid inertia is normally negligible (Stokes flow, *Re* ≪ 1).

Inertial microfluidics works in between Stokes and turbulent regimes (inertia and fluid viscosity are finite, ~1 < *Re* < ~100).

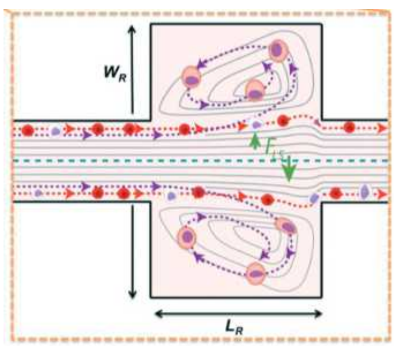


Particle sorting by inertial lift forces

J. Zhang et al., Fundamentals and applications of inertial microfluidics: a review, Lab Chip, 2016, 16, 10



Dean flow in curved channels



Vortices in expanding channels for cell trapping

Stokes flow and Stokes equations

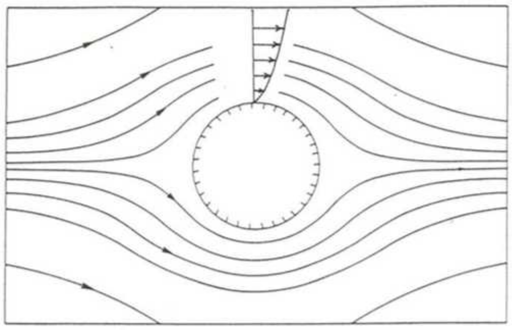
EPFL

Sir George Stokes, Bt.

Born	13 August 1819 Skreen, County Sligo, Ireland
Died	1 February 1903 (aged 83) Cambridge, England
Fields	Mathematics and physics
Institutions	University of Cambridge

Stokes, G. G.
On the Effect of the Internal Friction of Fluids on the Motion of Pendulums
Transactions of the Cambridge Philosophical Society, Vol. 9, pp. 8-93, 1851

He assumed that the flow is so slow that advective acceleration of the fluid as it passes around the sphere can be ignored, $(\mathbf{v}\cdot\nabla)\mathbf{v} = 0$.



The flow pattern is symmetrical front to back

Stokes flow and Stokes equations

EPFL

Stokes equation

$$0 = -\nabla p + \eta \nabla^2 \mathbf{v} \qquad (2.41)$$

Stokes flow or "creeping flow" (very slow !)

Significant simplification. Relevant for microfluidics !

A linear eqn with analytical solutions is some cases.

Reynolds introduced "his" number only in 1883, i.e. more than 30 after Stokes' intuitive approach.

For *Re* << 1 the non-linear term $\rho(\mathbf{v}\cdot\nabla)\mathbf{v}$ in the Navier-Stokes equation can indeed be neglected.

$$Re\left[\left(\tilde{\mathbf{v}}\cdot\tilde{\nabla}\right)\tilde{\mathbf{v}}\right] = -\tilde{\nabla}\tilde{p} + \tilde{\nabla}^2\tilde{\mathbf{v}}$$

Re < 0.1 is a good rule of thumb that the Stokes eqns are a good approximation of a real flow problem.

Stokes flow and Stokes equations

$$0 = -\nabla p + \eta \nabla^2 \mathbf{v} \qquad (2.41)$$

Expanded form + continuity eqn (no external forces, $\partial_t \mathbf{v} = 0$).

$$\frac{\partial p}{\partial x} = \eta\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

$$\frac{\partial p}{\partial y} = \eta\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right)$$

$$\frac{\partial p}{\partial z} = \eta\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right)$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

May be simplified for specific cases :

Taking the divergence ($\nabla \cdot$_) of (2.41) results in a Laplace equation for the pressure field $p(\mathbf{r})$, useful if boundary conditions are specified in terms of pressure.

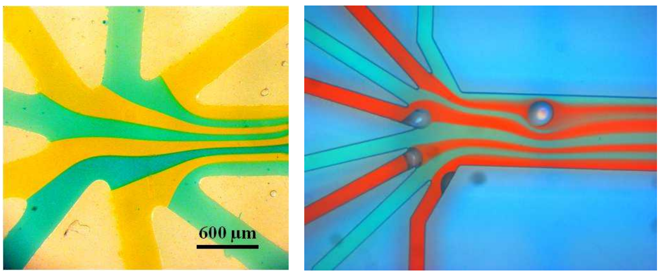
$$\nabla^2 p = 0 \qquad (2.45)$$

Taking the curl (∇ x _) of (2.41) results in a simple equation for the vorticity ($\boldsymbol{\omega} = \nabla \times \mathbf{v}$), useful if boundary conditions are specified in terms of velocitiy.

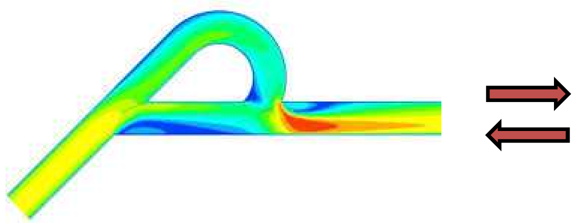
$$\nabla^2 \boldsymbol{\omega} = 0 \qquad (2.44)$$

Properties of Stokes flow

- Laminar flow pattern
- Linear in pressure and velocity (superposability of flow solutions).
- Instantaneity: No dependence on time except through time-dependent boundary conditions.
- Time reversibility of the flow. Flow symmetry around (before/after) an obstacle.
- Uniqueness ⇨ no flow instabilities.
- Minimum of dissipation of kinetic energy.



Microfluidic artwork showing laminar flow patterns.



Tesla valve. In the Stokes flow regime no "valving" effect is observed for inverted flow directions as forward and reverse flow paths.

see also Taylor–Couette flow in two rotating cylinders: https://www.youtube.com/watch?v=p08_KlTKP50

The transient form of the Stokes equation

$\partial_t \mathbf{v} \neq 0$ ⇨ **Transient form of the Stokes equation**

$$\rho \partial_t \mathbf{v} = -\nabla p + \eta \nabla^2 \mathbf{v}$$

Example: $\Delta p = 0$ for $t \geq 0$ (relaxing flow)

Stokes eqn takes the form of a **momentum diffusion equation** with the diffusion constant

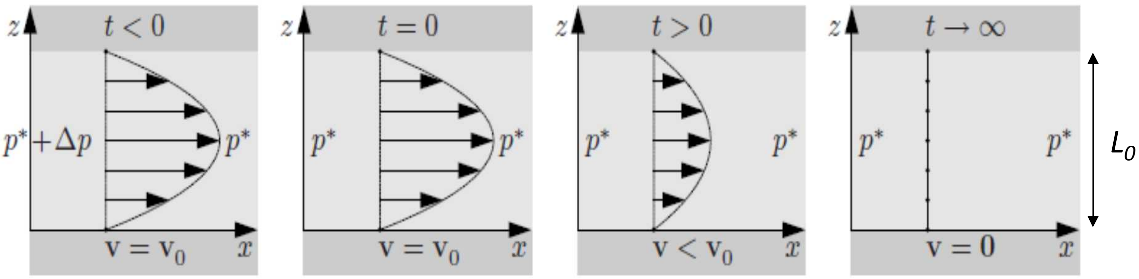
$\nu = \eta/\rho$ (kinematic viscosity [m²/s])

$$\partial_t \mathbf{v} = \nu \nabla^2 \mathbf{v}$$

Estimation of the time scale τ_0 to establish/or to stop a steady laminar flow upon application/release of an external pressure difference Δp. Balance of unsteady inertial and viscous force densities

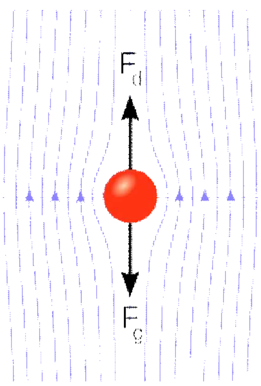
$$f_i \sim \frac{\rho V_0}{\tau_0} \qquad f_v \sim \frac{\eta V_0}{L_0^2} \qquad \Longrightarrow \tau_0 \sim \frac{\rho L_0^2}{\eta}$$

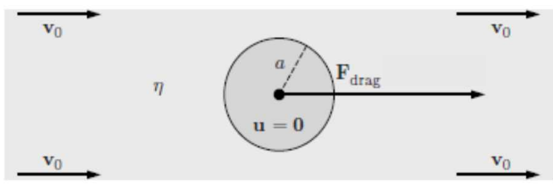
≈ 10 ms for a 100 μm channel



Unbounded Stokes flow around a sphere - Viscous drag force

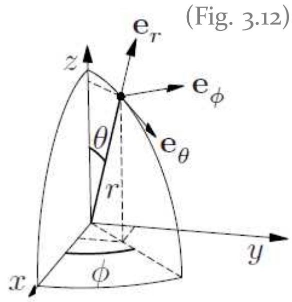
⇨ Determine the viscous force $\mathbf{F_d}$ (Stokes drag force) acting on a rigid sphere (microbead, radius a) moving with velocity v_0.





(Fig. 3.12)

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$



(Fig. C.2)

Stokes (1851) derived the **viscous drag force $\mathbf{F_d}$ on a sphere** by solving a simplified version of the Navier-Stokes eqns analytically.

"Stokes paradox": There is no non-trivial solution for the Stokes equations around an infinitely long cylinder.

Due to symmetry only the radial co-ordinate r and the polar angle θ enters in the calculation. The coordinate system may be chosen with the bead at rest and liquid flowing around.

Unbounded Stokes flow around a sphere

Creeping flow (*Re* <<1): Acceleration can be ignored / inertial forces $(\mathbf{v}\cdot\nabla)\mathbf{v} = 0$.
The fluid is further slowed down due to viscous forces when passing the bead surface.

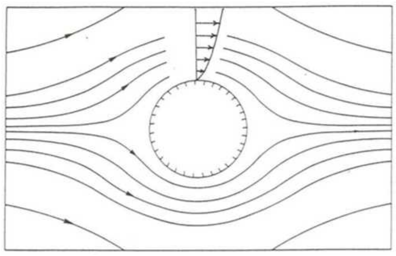
$$Re = \eta a V_0/\rho \ll 1$$

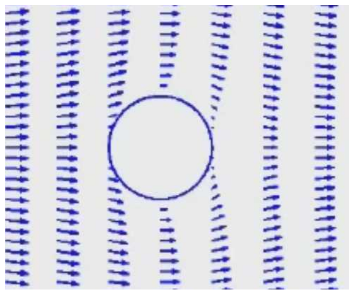
$$\nabla^2 \mathbf{v} = \frac{1}{\eta}\boldsymbol{\nabla} p$$

$$v_r = +V_0 \cos\theta \left[1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right]$$

$$v_\theta = -V_0 \sin\theta \left[1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right]$$

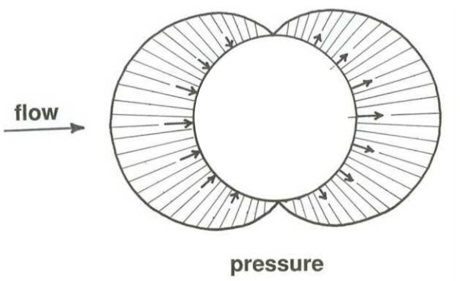
Velocity field $v(r,\theta)$ in terms of a power series in a/r. Boundary conditions: $v(a) = 0$ and $v(\infty) = v_0$





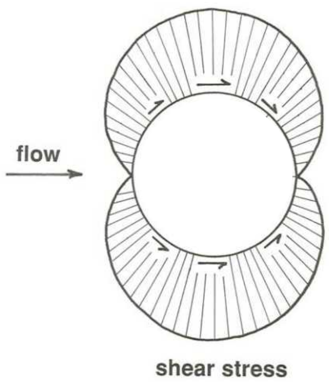
The flow pattern is symmetrical front to back

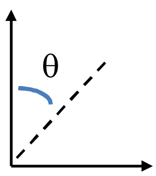
Unbounded Stokes flow around a sphere



$$p = p^* - \frac{3\eta V_0}{2a}\cos\theta$$

Pressure field on the sphere (radius a, p = ambient pressure)*





$$\sigma'_{\theta r} = -\frac{3\eta V_0}{2a}\sin\theta \qquad (3.126)$$

Shear stress on the surface of a sphere (radius a)

Unbounded Stokes flow around a sphere

EPFL

$$p = p^* - \frac{3\eta V_0}{2a}\cos\theta \qquad \sigma'_{\theta r} = -\frac{3\eta V_0}{2a}\sin\theta \qquad (3.126)$$

Pressure field on the sphere *Shear stress on a sphere*

⇨ The drag force $\mathbf{F}_d$ can be derived from the stress tensor $\boldsymbol{\sigma}$ as integral over the surface force densities (including the normal p components).

'Stokes Law' for the viscous drag force on a sphere

$$\boxed{F_{\text{drag}} = 6\pi\eta\, aV_0} \qquad (3.127)$$

accurate for $Re < 0.2$

Corrections:

Drag coefficient starts deviating for $\sim Re \geq 0.2$

$F_{drag} = 6\pi\eta a V_0\,(1 + 0.15 \cdot Re^{0.687})$

$0.2 < Re < 500$ -1000

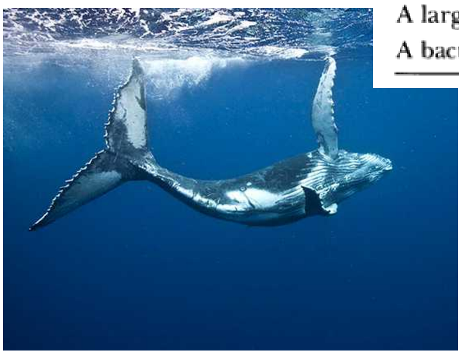
J. Zhang et al., Fundamentals and applications of inertia microfluidics: a review, Lab Chip, 2016, 16, 10

Drag on a sphere will be up to a factor 3 higher in the vicinity of a solid wall.

Life at small scale and low Re number

EPFL

	Reynolds Number
A large whale swimming at 10 m s⁻¹	300,000,000
A bacterium, swimming at 0.01 mm s⁻¹	0.00001



Propulsion mechanisms at high *Re*-numbers (humans, fish, etc.) are based on inertial effects, *e.g.* fins.

…but, inertia is totally irrelevant in the life of a microorganism, *i.e.* for swimming at low *Re*-number !

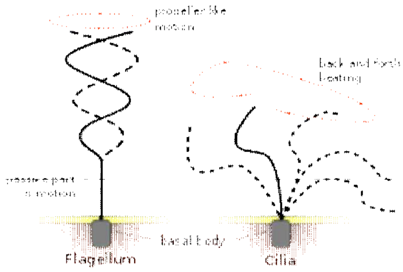
Example: A human swimming with **v ≈ 1 mm/h in honey** (L = 2m, η = 10 Pa·s, ρ = 1.5 kg/l) corresponds to $\boldsymbol{Re = 10^{-4}}$. A microorganism would probably feel like this.

Rubber band powered toy that tries to paddle forward viscous corn syrup.

At low *Re*-numbers any reciprocal motion (even if fast in one direction and slow in the return direction) does not result in forward motion due to the reversibility of the Stokes flow.

https://www.youtube.com/watch?time_continue=4&v=2kkfHj3LHeE

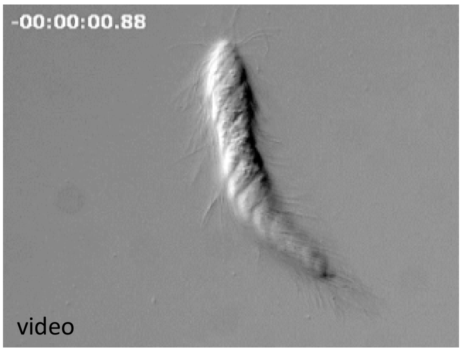
EPFL



Life in moving fluids: the physical biology of flow by S. Vogel (1996)

⇨ Microorganism have developed other propulsion mechanisms, such as flagella or cilia, working as a flexible oar or as a corkscrew.

Deforming the shape of the paddle breaks the symmetry of the stroke, creating more drag on the power stroke than on the recovery stroke.



PARAMECIUM (50 to 330 μm, an abundant genus of unicellular ciliates) covered with hair-like cilia.

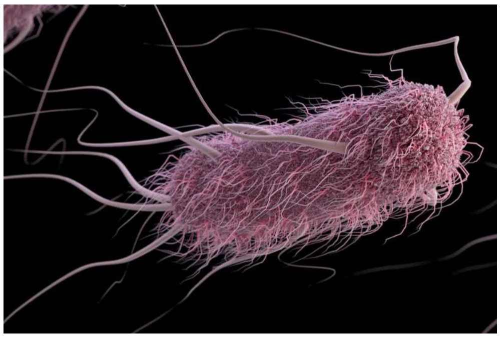


Illustration of an *Escherichia coli* based on a SEM micrograph. These bacteria use flagella for propulsion. (A. Eckert and J. Oosthuizen)

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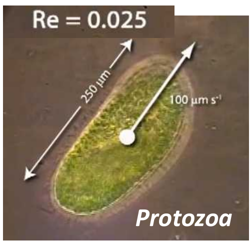
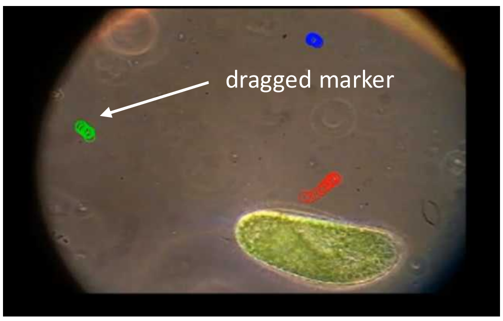
For microorganism swimming in water becomes very difficult, although the viscosity is very low (η = 1 mPa·s).

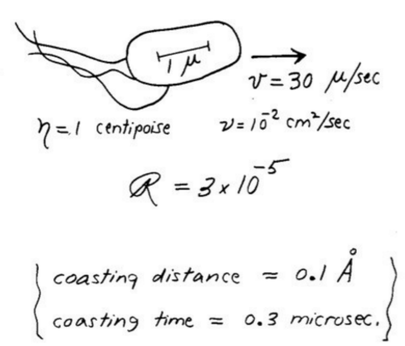




A protozoa (length 250 μm, 100 μm/s) drags water at a distances up to 250 μm (and more). Added mass ~200 μg, *i.e.* 100 x cell mass of ~2 μg

⇨ As if a human would swim with 10 tons attached to the feet !

http://www.youtube.com/watch?v=gZk2bMaqs1E



A bacteria typically moves at 20-40 μm/s. It takes him about 0.1 Å and 0.3 μs to stop.

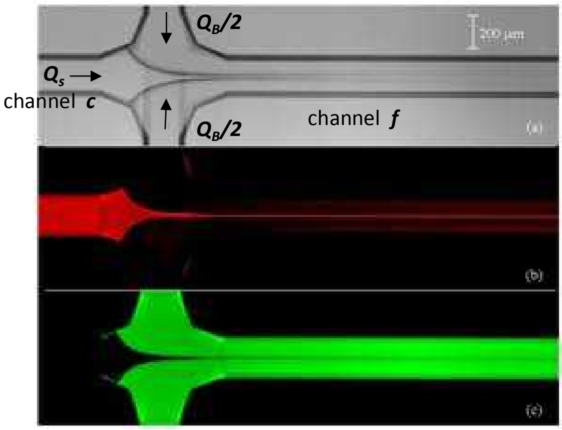
E.M. Purcell, *Life at low Re number,* America Journal of Physics, Vol. 45, p. 3-11 (1997)
Dusenbery, David B. (2009). *Living at Micro Scale*, Harvard University Press, Cambridge.

2.6 Hydrodynamic focusing

Example 1: 2D hydrodynamic focusing

A. Jahn, et al. , J. Am. Chem. Soc., 2004, 126 (9), 2674-2675

- Laminar flow conditions can be used to create well-defined fluidic interfaces, *e.g.* to focus fluid streams hydrodynamically.
- Principle: A central stream (w_{fs}, Q_s) is squeezed by a lateral sheath flow (2 x $Q_B/2$)
- Applications: Laminar mixers, cell cytometers, single molecule detection, etc.



2D hydrodynamic focusing *on a planar chip with two sheath flows.*
Silicon/glass microchannels: height 40 µm, width D_c=D_f =200 µm.
Total flow = 150 µL/min, $Q_b/Q_s \approx 20 \Rightarrow$ ***$w_{fs} \approx$ 10 µm****.*

Mass flow balance for Q_s:

$$Q_S = v_f w_{fs} h = v_c D_c h$$

Total flow balance in channel f:

$$Q_f = Q_S + Q_B = v_f D_f h$$

Width of focused steam:

$$w_{fs} = \left(\frac{D_f}{1 + \frac{Q_B}{Q_S}} \right)$$

Example 2: 3D hydrodynamic focusing using Dean flow

- On-chip 3D focusing combining laminar sheath flow and inertial effects.
- Re number must be high enough to allow for circulating secondary flow (~1 < *Re* < ~100).
- Simplifies fluidic design and control, but high flow speeds !

Dean flow

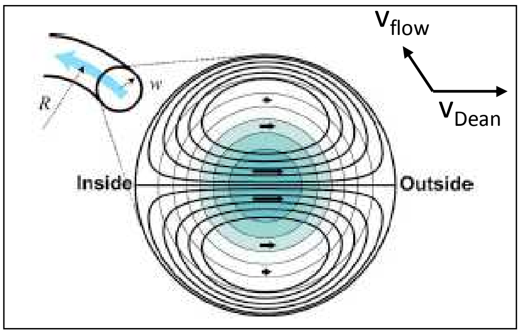
Inhomogeneous flow profile causes centrifugal forces driving a circulating flow in slightly curved channels ($R >> w$).

The magnitude of the centrifugal force density is greatest in the center, where the primary flow is fastest: $f_i \approx V_0^2[1-(r/w)^2]^2/R$

Stable solution with a pair of vortices occur for low **Dean numbers** (< 950).

$$De \approx Re\,(w/2R)^{0.5}$$

w is the of the tube diameter or channel width, R is the radius of curvature of the path of the channel.



T. M. Squires et al, 2005, Rev. Mod. Phys., ***77****, 977-1026*

(see also Chapter 4.2.5: "*A multivortex mixer based on inertial flow properties*")

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Example of a single-layer planar device (PDMS)

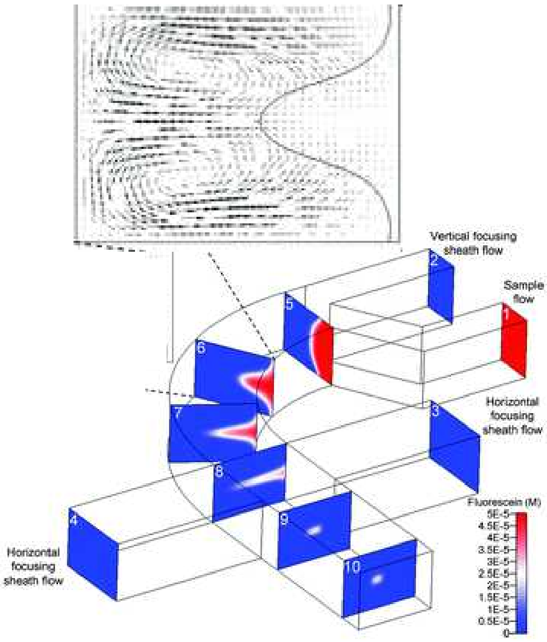
The secondary flow velocity field shows strong Dean vortices.

⇨ “Microfluidic drifting” resulting in stretching of the sample flow acroos the channel width (vertical focusing, red).

⇨ Two lateral sheath flows are introduced for horizontal focusing.

Re = 74 (!), *De* ≈ 43

High flow speed in the range of ≈ m/s !



Cross-sectional profiles of the fluorescein dye concentration in the focusing device. Inset: simulation of the secondary flow velocity field shows Dean vortices in the 90° curve.

Channel w = 100 µm, h = 75 µm, L = 1cm, R_{curve} = 250 µm

X. Mao et al., Lab Chip, 2007, 7, 1260-1262

X. Mao et al., Lab Chip, 2009, 9, 1583-1589

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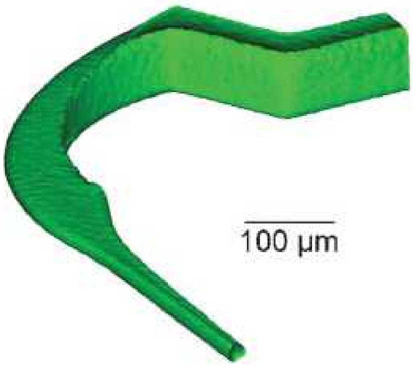
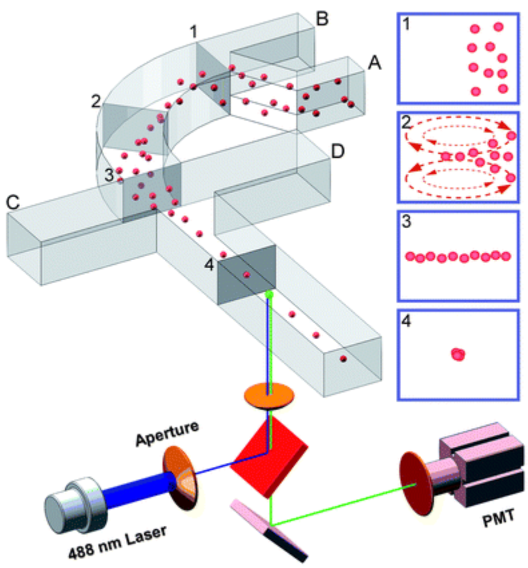


Focused beam ∅ ≤ 15 µm

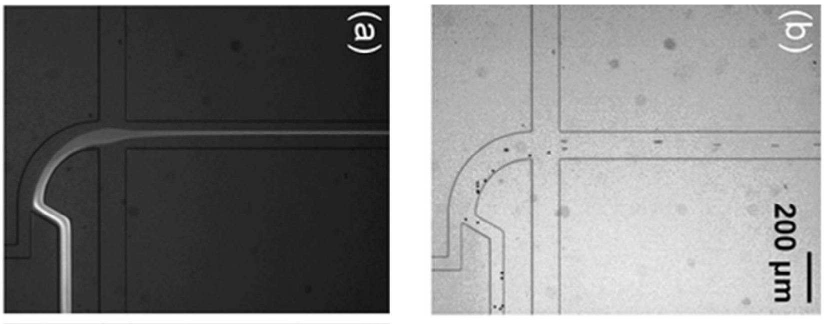
The 3D architecture of the sample flow during the focusing process characterized by confocal microscopy (fluorescein solution).



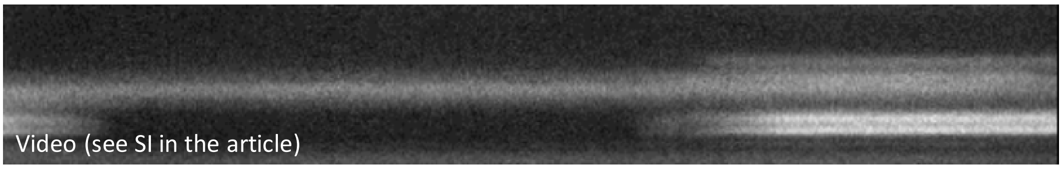
Inlet A: Cells or particles; Inlet B: vertical focusing sheath flow; Inlets C and D: horizontal focusing sheath flows. Inset 2 represent the Dean vortices. Laser-induced fluorescent detection is shown.

X. Mao et al., Lab Chip, 2007, 7, 1260-1262

3D focusing for on-chip cytometry



(a) Fluorescent and (b) bright-field top view of the particle flow (∅ 8 µm).



Side view of the particle flow with focusing height ≈12 µm: Particle velocity 3.6 m s^{-1}, 1700 particles/s^{-1}

X. Mao et al., Lab Chip, 2007, 7, 1260-1262